Single-Particle Electrodynamics

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30 June 1994

Thesis submitted for the degree of

Doctor of Philosophy

at the University of Melbourne (School of Physics)

This thesis is less than 100,000 words in length

Produced on acid-free paper

Abstract

This thesis addresses various issues in relativistic electrodynamical frameworks describing systems containing only a *single particle*—namely, classical electrodynamics, and single-particle relativistic quantum mechanics. The main emphasis is on clearing up several outstanding problems in *the classical theory of pointlike particles carrying dipole moments*; the Dirac equation also considered, but only to verify that particular results are in accord with the quantum theory.

There are three major results presented in this thesis:

Firstly, the fully relativistic classical equations of motion are obtained for a point particle carrying electric charge and electric and magnetic dipole moments, ignoring the effects of radiation reaction. Most of the terms of the nonrelativistic limits of these equations have been been suggested previously, but one term of subtle origin is new. Furthermore, some of the physical arguments used by previous authors to explain the deviation of the magnetic dipole force from the standard textbook result have been incorrect; these issues are discussed and clarified.

Secondly, simple yet explicit expressions are obtained for the retarded classical electromagnetic fields, at arbitrary positions in space, generated by a point particle carrying electric charge and electric and magnetic dipole moments, in arbitrary relativistic motion. The manifestly covariant field expressions have been obtained previously, but their simple and explicit expression is, to the author's knowledge, new. The results are comprehensively verified by means of a computer algebra program written for the purpose.

Thirdly, the preceding results are brought together to obtain the complete classical radiation reaction equations of motion for point particles carrying electric charge and electric and magnetic dipole moments, in arbitrary relativistic motion. Such an analysis has, to the author's knowledge, only been attempted twice before, under differing sets of assumptions; the fully classical results herein are presented in terms of concepts that have been found to be of most use in modern electrodynamical applications. The results are applied to the specific example of reaction due to spontaneous M1 magnetic dipole radiation; the successes and limitations of the completely classical analysis are discussed.

Copyright Declaration

Acknowledgments

This thesis describes work carried out at the School of Physics of the University of Melbourne, from 1 January 1991 through 30 June 1994, towards the degree of Doctor of Philosophy. This work was financially supported, throughout this period, by an Australian Postgraduate Research Allowance; and additionally, for the first twelve months, by a Dixson Research Scholarship. It was also supported indirectly by the Australian Research Council, via the resources of the Theoretical Particle Physics Group of the School of Physics of the University of Melbourne.

Some of this work was carried out during a two-month visit, in early 1992, to the United States Department of Energy's Institute for Nuclear Theory at the University of Washington, which is supported by US-DOE Grant #DOE/ER40561. I would like to offer my heartfelt thanks to all at the Institute for Nuclear Theory for their generosity and helpfulness. The following institutions also provided generous support for technical visits to their establishments in 1992: Los Alamos National Laboratory; CEBAF; the University of California at San Diego; and TRIUMF.

I would like to thank the following people for innumerable discussions and unending assistance throughout my studies: J. Anandan, D. P. Barber, S. D. Bass, M. V. Berry, J. Boatman, L. A. Bursill, J. H. Choi, G. J. Clerk, J. A. Daicic, A. J. Davies, C. P. Dettman, J. Filshie, N. E. Frankel, D. Fryberger, W. Haxton, X.-G. He, P. Herczeg, K. C. Hines, L. L. Hollenberg, B. Holstein, V. D. Hopper, K. Jones, G. C. Joshi, I. B. Khriplovich, S. A. Karataglidis, A. G. Klein, K. Lekamge, J. Millener, J. M. Namyslowski, M. Narielvala, K. A. Nugent, G. I. Opat, V. S. Pantelidis, J. Pasupathy, W. Rowlands, E. Smart, J. Spencer, G. J. Stephenson Jr., M. J. Thomson, S. N. Tovey, L. Wilets, L. Wolfenstein, D.-D. Wu, T. Zier, D. A. Zubanov.

I would like to thank Peter O'Brien for putting up with my frequent cries for computational help and violations of disk-space limits.

I apologise to Dr. Geoff Taylor for never submitting the final write-up of my first and only experimental physics project, in 1989, and promise to recite the Particle Data Book summary tables as repentance.

I would particularly like to thank Assoc. Prof. Bill Wignall for agreeing to talk some Quantum Mechanics with a bright-eyed student in 1987, and for regularly deflecting me onto topics of interest and importance ever since.

I thank Veronica Pantelidis for her moral support, from the other side of the planet, and I hope I may meet her and the two Nicks in person some day.

I thank Henry and Miriam Cieslak for being such excellent neighbours, and especially for knocking on my door with plates of food, in the past few weeks, when I forgot to eat.

I thank Russell Petty for keeping me sane!

To my Ph.D. supervisor, Professor Bruce McKellar, I owe an unrepayable debt, for agreeing, in 1990, to supervise an unknown Engineering student who had not even taken the prerequisite undergraduate course in Quantum Mechanics. On reading this thesis, I see the products of Bruce's guidance on every page, and am reminded of the many long but rewarding afternoons on which we struggled to unlock Nature's secrets.

I must thank my various girlfriends, during this time, for reminding me that relationships are not analytical functions of time: no matter how many derivatives you measure, you never know what's going to happen next.

The lifelong friendship of Roger Behrend, and the mathematical guidance he has provided, is most deeply appreciated.

Most of all, I would like to thank my Mum, for being the best Mum in the world.

This work has serious difficulties in it. There are many quantities appearing in the theory which turn out to be infinite although they ought to be finite; and physicists have followed all sorts of tricks to avoid these difficulties, but the result is that the theory is in rather a mess. The departures from logic are very serious and one really gives up all pretense of logical development in places. I propose to follow different lines, which avoid some of the difficulties. I won't be able to avoid all of them, but I will avoid some of the worst ones and I will, at any rate, be able to preserve some semblance of logic. By that I mean that I will neglect only quantities that one can have some physical feeling to count as small, instead of doing what people often do in this kind of work: neglect something that is really infinitely great. It will not be possible for me to set things out rigorously. I shall neglect things I shan't be able to prove are really small, but still I hope you'll have some feeling that these things are small.

The system of approximations I shall use will be somewhat similar to the approximation that engineers use in their calculations. Engineers have to get results and there are so many factors occurring in their problems that they have to neglect an awful lot of them; they don't have time to study everything seriously and they develop a sort of feeling as to what can be neglected and what can't. I believe that physicists will have to develop a similar sort of feeling as to what can be neglected and what can't. The final test is whether the resulting theory is coherent and in reasonable agreement with experiment.

—— P. A. M. Dirac (1966),
electrical engineer and physicist,
on renormalisation [70].

Contents

Chapter 1

Overview of this Thesis

> *My interest in classical electromagnetism has waxed and waned, but never fallen to zero. The subject is ever fresh.*
>
> —— J. D. Jackson (1974) [113]

1.1 Introduction

Most physicists do not share Jackson's relative enthusiasm. Not only is interest in classical electrodynamics not generally positive-definite, it frequently crosses the axis altogether: the antipathy can be palpable.

Classical electrodynamics is not "trendy". It is, in the minds of many, a dead subject—a vein that was mined out long ago. Its reputation is not helped by the disturbing tendency for "new" results to be so revolutionary as to be incredible—literally. Many seasoned veterans of the field can relate to Feynman's description [86] of another area of classical physics, since improved:

> I am getting nothing out of the meeting. I am learning nothing. Because there are no experiments this field is not an active one, so few of the best men are doing work in it. The result is that there are hosts of dopes here and it is not good for my

> blood pressure: such inane things are said and seriously discussed that I get into arguments outside the formal sessions (say, at lunch) whenever anyone asks me a question or starts to tell me about his "work". The "work" is always: (1) completely ununderstandable, (2) vague and indefinite, (3) something correct that is obvious and self-evident, but worked out by a long and difficult analysis, and presented as an important discovery, or (4) a claim based on the stupidity of the author that some obvious and correct fact, accepted and checked for years, is, in fact, false (these are the worst: no argument will convince the idiot), (5) an attempt to do something probably impossible, but certainly of no utility, which, it is finally revealed at the end, fails, or (6) just plain wrong. There is a great deal of "activity in the field" these days, but this "activity" is mainly in showing that the previous "activity" of somebody else resulted in an error or in nothing useful or in something promising. It is like a lot of worms trying to get out of a bottle by crawling all over each other. It is not that the subject is hard; it is that the good men are occupied elsewhere.

Given the frequent disappointments, it is all the more rewarding when a new piece of classical physics shows itself to be a real gem. Unfortunately, they are just as rare.

The author, being young and enthusiastic, has generated numerous "discoveries" falling into one or more of Feynman's six categories. It is thanks to those of wiser years, listed explicitly in the Acknowledgments, who have weathered the author's pronunciations on these revelations, that the worst violations of common sense have been eradicated; the most suspect assumptions redrafted; the most dubious arguments rigourised; and the elegance of the resulting conclusions buffed to a high polish. The inevitable inaccuracies and oversights surviving this process, both in form and in substance, the author claims ownership for.

It must be noted, from the outset, that the author is an engineer, not a physicist or mathematician. The author's prime objective is *to obtain the correct mathematical description of the real physical world*. Elegance of such a correct description is, by the very nature of Nature, assured; but elegance in the particular path travelled by the author during this quest is of no import: let others, more capable, derive the results contained herein on the back of an envelope, by means of wonderfully beautiful techniques, secure in the knowledge that the author's cumbersome yet intuitively understandable derivation rests safely on their shelves.

It should also be noted that the results obtained in this thesis are not revolutionary. No well-known law of physics is challenged. Some generally accepted answers, of sometimes dubious origin, are put on a somewhat more secure footing—or, at the very least, are given several additional dubious origins. Some existing results that are mathematically undeniable, but somewhat perplexing intuitively, are viewed from a new direction, or in some instances from several new directions; whether the eye of the beholder approves is beyond the author's control. Some previously incomplete lines of investigation are taken to their logical conclusion; the result is often surprisingly simpler than the existing terminus. And finally there are the completely new results, obtained after the traversing of much uncharted territory, which may or may not be ultimately accepted as physically correct, but which, for the moment, are the only candidates we have available to us.

If the reader finds, among these investigations, just a single paragraph that raises the mist of one's understanding of the physical world, then the author has succeeded.

1.2 What do I need to read?

If the reader wishes to find a particular result, or set of results, rumoured to be contained within this thesis, then they should consult either the Table of Contents, or the more detailed descriptions in the sections following this one, to pinpoint the topic or topics of interest to them. They should also, however, cast a glance at Appendix A, which describes the notations, conventions, *etc.*, that the author has used throughout this thesis.

If, on the other hand, the reader wishes to read the entire thesis, then they should proceed with the five non-appendical chapters following this one. Each chapter draws on results established in the previous chapters, as a general rule; but each may also be considered separately, if one is willing to accept the preceding results, sight unseen. Chapters 2 and 3 review the formalism that will be used by the author; no major new results are presented; and the style is somewhat more didactic than that of the chapters that follow. The major new results found by the author are presented in Chapters 4, 5 and 6, each of which deals with one of the three problems listed in the Abstract of this thesis.

The notations and conventions listed in Appendix A should be consulted as and when the need arises; the material therein has been compiled and written most carefully, to avoid ambiguity or confusion; but as a result the style of the text is necessarily more clinical and pedantic than the discussions contained in the other chapters of this thesis.

Material relegated to the appendices may or may not be of interest, but in any case is not required reading. Some readers may be interested in the computer algebra programs written by the author, whose output is contained in Appendix G, which were used to verify and complete the algebraic considerations of this thesis. Due to length restrictions, the corresponding ANSI C source code files are not listed in this thesis; they are, however, contained in digital copies of this thesis (*see* Section A.2.2).

The following sections summarise the remaining chapters and appendices contained in this thesis.

Chapter 2: Classical Particle Mechanics

Section 2.1: Introduction

Chapter 2 briefly reviews the various formulations of classical particle mechanics.

Section 2.2: Point particles

The notion of a *point particle* is discussed.

Section 2.3: Newtonian mechanics

Newtonian mechanics is reviewed, with an emphasis placed on those aspects of the formalism that are frequently confused in the literature.

Section 2.4: Lagrangian mechanics

A similar review is provided of the Lagrangian and Hamiltonian formulations of classical mechanics.

Section 2.5: The canonical–mechanical challenge

The author challenges the reader to discover how many times mechanical and canonical quantities are confused in the volumes sitting on their bookshelves.

Section 2.6: Relativistic mechanics

The notations and quantities used by the author for relativistic analyses are briefly introduced. Subtleties of the use of relativistic kinematics are commented on.

Section 2.7: Classical limit of quantum mechanics

The extent to which classical results are the limit of the corresponding quantum mechanical analyses is briefly discussed.

Section 2.8: Pointlike trajectory parametrisation

Quantitative expressions are found, in a given Lorentz frame, for the fully relativistic trajectory of a point particle around the instant that it is at rest.

Chapter 3: Relativistically Rigid Bodies

Section 3.1: Introduction

Chapter 3 discusses the problems involved in defining rigidity in the presence of relativistic requirements, and shows that, with suitable care, they are not insurmountable.

Section 3.2: Notions of rigidity

The Galilean and Einsteinian notions of rigidity are reviewed.

Section 3.3: Trajectories of rigid body constituents

The notion of a "constituent" is discussed. Quantitative expressions are obtained for the trajectories of the constituents of a relativistically rigid body. The accelerative redshift factor is derived and discussed. The spin degrees of freedom for the constituents of the rigid body are also defined.

Chapter 4: Dipole Equations of Motion

Section 4.1: Introduction

Chapter 4 obtains the classical relativistic equations of motion for pointlike particles carrying electric charge and electric and magnetic dipole moments. Radiation reaction is not included at this stage; this is investigated in Chapter 6.

Section 4.2: Newtonian mechanics

Electric and magnetic dipoles are analysed using Newtonian mechanics.

Section 4.3: Lagrangian mechanics

Electric and magnetic dipoles are analysed using Lagrangian mechanics.

Section 4.4: What does the Dirac equation say?

The Dirac equation is analysed, in the Newton–Wigner representation, for both a pure Dirac and an anomalous Pauli magnetic moment, and the resulting equations of motion compared to those found in the preceding sections.

Chapter 5: The Retarded Fields

Section 5.1: Introduction

Chapter 5 obtains both manifestly covariant and explicit expressions for the retarded electromagnetic fields generated by pointlike particles carrying electric charge and electric and magnetic dipole moments.

Section 5.2: History of the retarded dipole fields

A brief historical review is given of the work undertaken by previous authors to obtain the retarded dipole fields. The author's motivation, and rôle in the saga, are noted.

Section 5.3: The Liénard–Wiechert fields

A review is provided of the derivation of the standard Liénard–Wiechert electromagnetic fields generated by a point charge in arbitrary motion. Concepts, techniques and notation to be generalised in the next section to the case of dipole moments are emphasised.

Section 5.4: The retarded dipole fields

The retarded fields from a point particle carrying electric and magnetic dipole moments are derived. The manifestly-covariant results are reëxpressed in explicit, non-covariant notation, using several new convenient quantities introduced by the author; the results are surprisingly simple.

Section 5.5: The static fields

The static fields generated by a spherical body of uniform charge and polarisation density are reviewed, with two aspects particularly emphasised: the behaviour of the fields in the vicinity of the source volume; and the mechanical self-energy, mechanical self-momentum and mechanical self-angular momentum contained within the fields.

The extra Maxwell contribution to the point magnetic dipole field is also obtained, in manifestly-covariant form, for arbitrary relativistic motion of the dipole.

Chapter 6: Radiation Reaction

Section 6.1: Introduction

Chapter 6 obtains the radiation reaction equations of motion for a pointlike particle carrying electric charge and electric and magnetic dipole moments.

Section 6.2: Previous analyses

The previous attacks on this problem by Bhabha and Corben, and Barut and Unal, are briefly reviewed.

Section 6.3: Infinitesimal rigid bodies

Various issues arising when the relativistically rigid bodies of Chapter 3 are of *infinitesimal size* are described.

Section 6.4: The sum and difference variables

The sum and difference constituent variables are introduced, and the necessary mathematical groundwork for their use developed.

Section 6.5: Retarded kinematical quantities

The kinematical quantities of the generating constituents are evaluated at their respective retarded times, and from these quantites the self-fields are computed.

Section 6.6: Divergence of the point particle fields

The three-divergences of the fields of a *point* particle in arbitrary motion are evaluated, around the worldline of the particle, with particular attention given to the electromagnetic moment *source terms* implied by these divergences via the Maxwell equations.

Section 6.7: Inverse-cube integrals

Divergent integrals in $\boldsymbol{r}_d$-space are regularised using the methods of the previous section. The logarithmically divergent terms are identified and labelled, allowing the finite contributions to the equations of motion to be extracted.

Section 6.8: Computation of the self-interactions

All of the results of this thesis are finally brought together to obtain the radiation reaction equations of motion for point particles carrying electric charge and electric and magnetic dipole moments.

Section 6.9: Discussion of the final equations

The final equations of motion of the previous section are discussed.

Section 6.10: Sokolov–Ternov and related effects

The Sokolov–Ternov and Ternov–Bagrov–Khapaev effects (spin-flip due to emitted radiation) are briefly reviewed, and a subset of the author's equations are used to analyse the phenomena classically. The successes and limitations of the completely classical analysis are brought into stark relief.

Appendix A: Notation and Conventions

Section A.1: Introduction

Appendix A describes in detail all the notational and other conventions followed throughout this thesis, and should be glanced at before one reads the body of this thesis.

Section A.2: Physical format of this thesis

It is explained that this thesis is available in both paper and digital formats.

A description is given of the files included in digital copies of this thesis, and instructions are given for the digital processing of the whole thesis, or any single chapter or appendix.

The computer algebra programs included in digital copies of this thesis are described in more detail in Appendix G.

Section A.3: Language and typography

Descriptions are given of the document preparation system used to prepare this thesis, the spelling and punctuation choices made, the system of units used, and the symbols used to represent quantities.

Special or ambiguous notation or nomenclature, that is either of a universal nature or is not specifically mentioned elsewhere, is defined here.

Section A.4: Enumeration sets

The terminology surrounding *enumerations* is borrowed from the field of computer programming. The extra nomenclature serves to simplify the description of other mathematical operations in the following sections.

Section A.5: Special functions

Notation and physical interpretation is supplied for the Kronecker and Dirac delta functions, and the Heaviside step function. The sign convention used in this thesis for the alternating function is specified.

Section A.6: Associativity and commutativity

Various functions and pieces of nomenclature are established to deal with the associativity and commutativity of quantities.

Section A.7: Matrices

Notation and nomenclature are defined for matrix quantities.

Section A.8: Relativistic mechanics

Various issues surrounding the use of relativistic mechanics are addressed.

A substantial amount of nomenclature is given a precise definition for the purposes of this thesis. Notation for Lorentz-covariant quantities is described in detail, as are the operations between them, and the notation used to denote such operations.

Notation for the kinematical quantities of a point particle's motion is established. Particular attention is given to the notation for the proper-time derivatives of kinematical quantities, for which there are two subtly different definitions (*see* Chapter 2).

Section A.9: Euclidean three-space

As with the previous section, a substantial amount of nomenclature is given a precise definition, and notations are specified for various operations on three-vectors.

Appendix B: Supplementary Identities

Section B.1: Introduction

Appendix B is a convenient collection of various mathematical identities that are used throughout this thesis, that are either of common knowledge or can be derived simply. Identities that require substantial argument on the part of the author to establish their veracity are derived elsewhere in this thesis.

Section B.2: The electromagnetic field

The definition of the electromagnetic field strength tensor $F(x)$ in terms of the four-potential $A(x)$ is given. The Maxwell equations are listed in various forms. The dual field strength tensor $\widetilde{F}(x)$ is defined. Explicit components are listed for F and $\widetilde{F}$.

Various quadratic identities involving products of F and $\widetilde{F}$, that are not commonly known, are listed.

Section B.3: Three-vectors

Various identities are listed for three-vector operations. These are a copy of those given in the front cover of Jackson [113], and are listed here only for convenience.

Section B.4: Alternating functions

The connection between the three- and four-dimensional alternating functions, according to the sign convention established in Section A.5.4, is listed. The contractions of two four-dimensional alternating functions over one, two, three and four indices are given explicitly.

Section B.5: Four-vector cross-product

Explicit components for the four-vector cross-product operation, defined in Section A.8.10, are given. Cyclic invariance of the four-cross-product is illustrated.

Section B.6: Radiation reaction gradients

Various explicit expressions are given for the gradients of field expressions in some given direction. These are used for the gradient-force radiation reaction calculations in Chapter 6.

Appendix C: Supplementary Proofs

Section C.1: Introduction

Appendix C contains various explicit proofs that are not of sufficient importance to be included in the body of the thesis, but whose inclusion may be of use to some readers.

Section C.2: Mechanical field excesses

The excess in mechanical energy and mechanical momentum of the electromagnetic field due to the bringing of an electric charge into the vicinity of other electric sources are computed.

Section C.3: Electromagnetic field identities

Proofs are supplied for various identities listed in Section B.2.12.

Appendix D: Retarded Fields Verification

Section D.1: Introduction

Appendix D shows that the manifestly covariant retarded dipole fields obtained by the author in Chapter 5 agree with the expressions previously obtained by Cohn and Wiebe in 1976.

Section D.2: The Cohn–Wiebe field expressions

The final potential and field expressions obtained by Cohn and Wiebe are listed, together with comments about their conventions and notation.

Section D.3: Conversion of conventions

The Cohn–Wiebe potential and field expressions are transformed to take into account the differences between their basic conventions, and those used in this thesis.

Section D.4: Conversion of notation

The notation used for the quantities appearing in the potential and field expressions of the previous section is converted to the notation used for the equivalent quantities in this thesis.

Section D.5: Verification of the retarded potentials

It is shown that the Kolsrud–Leer four-potential agrees with that extractable from the author's derivation in Chapter 5.

Section D.6: Verification of the retarded fields

The Cohn–Wiebe field expressions are converted into the notation used in this thesis. It is shown that results obtained by the author in Chapter 5 agree with the converted Cohn–Wiebe expressions.

Appendix E: The Interaction Lagrangian

Section E.1: Introduction

Appendix E provides a brief derivation of the most general electromagnetic interaction Lagrangian possible for a spin-half particle, and obtains its classical limit.

Section E.2: Quantum field theory

The most general electromagnetic interaction vertex possible for a spin-half particle is obtained from the point of view of quantum field theory, and reduced to its simplest form.

Section E.3: The classical limit

The Lagrangian corresponding to the general interaction vertex function is obtained, in the classical limit, and the electromagnetic moments identified.

Section E.4: Source-free regions

The classical interaction Lagrangian is examined in source-free regions; it is found that the equations of motion derived therefrom simplify considerably.

Appendix F: Published Paper

Section F.1: Introduction

Appendix F contains a verbatim copy of the published paper that arose from the work described in Chapter 4.

Appendix G: Computer Algebra

Section G.1: Introduction

Appendix G contains all of the information about the computer algebra programs written by the author to compute and verify the algebraic results contained in this thesis.

Section G.2: Description of the programs

Descriptions are given of each of the five computer algebra programs written by the author: what they do, how they do it, and why they were written in the first place.

Section G.3: Running the programs

Detailed instructions are given for running the five ANSI C programs written by the author, and viewing the output, on one's local computer system. (Needless to say, the programs are only included in digital copies of this thesis.)

Section G.4: KINEMATS: Kinematical quantities

Herein is the full-LaTeXed output of the program KINEMATS, which computes various kinematical quantities required in this thesis.

This section serves as an appendix in its own right: much reference material is contained here that is not listed elsewhere. The fact that this section was in fact written by a computer, and not a human, should be ignored: the author has "written through" the C code with explanatory text, so as to make the output understandable.

(To watch this section being written before your very eyes, refer to Section G.3.)

Section G.5: RETFIELD: Retarded fields

As with the previous section, this section contains the fully LaTeXed output of the program RETFIELD, which verifies the explicit retarded field expressions obtained by the author in Chapter 5.

The material in this section is not, however, of great reference value—the main purpose of the program being the verification of manually-computed results, not the establishment of new results.

Section G.6: RADREACT: Radiation reaction

This section contains the fully LaTeXed output of the program RADREACT, the most complex program of the five written by the author for this thesis. It verifies the pointlike particle trajectory parametrisation of Chapter 2, and the relativistically rigid body constituent trajectories of Chapter 3. It then uses the results of the program RETFIELD to compute the self-fields generated by an infinitesimal relativistically rigid body. From these expressions, the radiation reaction self-interaction equations of motion are obtained.

As with Section G.4, the contents of this section serve as an appendix in their own right: much material contained here is not fully listed anywhere else.

Again, although this section was written by a computer, the author's parenthood is manifest.

Section G.7: TEST3INT: Testing of 3-d integrations

This section contains a copy of the ASCII text output of the program TEST3INT, which verifies explicitly the correct functioning of the 3-d integration routines used in the program RADREACT.

Section G.8: CHECKRS: Checking of inner integrals

This section contains a copy of the ASCII text output of the small program CHECKRS, which performs a rudimentary numerical integration used to explicitly verify the integrity of the inner ($\boldsymbol{r}_s$) integral results, derived analytically in Chapter 6.

Bibliography

As with any area of classical physics, the compilation of an exhaustive bibliography is probably not only impossible, but also futile. The Bibliography included by the author in this thesis is therefore a compromise between utility and practicality.

The works that the author has used as a basis for his investigations are listed without fail. Important works with a direct bearing on or relevance to the topics covered in this thesis are also included, especially where they themselves contain an important list of references. Works of only subsidiary or passing importance are sometimes included, but sometimes not, depending on the whims of the author; if a number of works cover essentially the same ground, a subset may be selected as more suiting the author's tastes than the others. Textbooks are referred to where necessary, but the author has not felt the need to compare the merits of the coverage of every topic in every textbook available; in practice, Jackson [113] and Goldstein [96] are used by default, except where a particular topic is covered in noticeably more depth elsewhere.

Of course, there always exists the possibility, despite vigorous efforts to the contrary, that entire lines of investigation by previous workers, into the topics covered in this thesis, have evaded the author's detection altogether; the sheer volume of literature covering the decades of relevance to this thesis, together with the lack of computerised search systems for such ageing tomes, makes this a serious concern. If any such major works *do* exist in the literature, somewhere, then the author may confidently state that they are not well-known—or, at any rate, not well-cited—by contemporary workers; and hence it may reasonably be argued that any controversy that may erupt over the author's omission of such a work from his Bibliography would be quite a healthy contribution to the current state of awareness of the physics community. Regardless, it is hoped that any omissions, major or minor, will be forgiven by their respective authors (where still living).

To make the Bibliography most user-friendly, in its paper form, the entries are sorted alphabetically on the surname of the first author, and then by year of publication; but also, for each and every co-author, a cross-reference entry has been inserted. (This only applies if there are five or less authors; if there are more than five, only the first is listed, and "*et al.*" appended; there are no cross-references in this case.) Thus, for example, if one recalls that Telegdi wrote an important paper on spin precession, but one cannot recall the reference, then the entry Telegdi (1959) directs the reader to the Bargmann–Michel–Telegdi paper, without one having to recall that Bargmann was the first author. This means that the total number of entries in the Bibliography is actually the sum of the products of the number of publications contained therein by the number of authors of each publication—rather than simply the number of publications, as is usually the case; it is felt that the convenience of this method to the reader outweighs the additional few pages it adds to the length of this thesis.

Now, let the Games begin.

Chapter 2

Classical Particle Mechanics

I derive from the celestial phenomena the forces of gravity with which bodies tend to the sun and the several planets. Then from these forces, by other propositions which are also mathematical, I deduce the motion of the planets, the comets, the moon, and the sea. I wish we could derive the rest of the phenomena of Nature by the same kind of reasoning from mechanical principles, for I am induced by many reasons to suspect they all may depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards one another, and cohere in regular figures, or are repelled and recede from one another. These forces being unknown, philosophers have hitherto attempted to search Nature in vain; but I hope the principles laid down here will afford some light either to this or some truer method of philosophy.

—— I. Newton (1686) [158]

2.1 Introduction

Three hundred and eight years later, a truer method of philosophy has not yet become apparent. We still understand Nature by tearing it into smaller and smaller pieces, and trying to understand the interactions between the most fundamental constituents we find.

To date, we have not been able to tear the *electron* into smaller pieces. The *muon* and *tauon* decay, but appear to be as structureless as the electron. The *proton* and *neutron* clearly have structure—arguably described quite well by quarks and gluons,—but may in many situations be considered approximately "elementary".

These are the "particles" the author has in mind when talking of "Single-Particle Electrodynamics"—although they will rarely be referred to by name again. Rather, this thesis considers the question of obtaining the *correct theoretical equations of motion*, according to classical electrodynamics, for idealised pointlike particles, carrying various electromagnetic moments. Although practical applications are at all times uppermost in the author's mind, the explicit consideration of such applications lies, in general, outside the scope of this thesis.

In this chapter, we review various aspects of classical particle mechanics that will be used in the remainder of this thesis. Emphasis is placed on those aspects of the mathematical framework that are, in the view of the author, either contentious, not widely appreciated, or cumbersomely formulated; results that are well-known, and not under challenge, are simply listed for convenience—the reader being referred to standard texts (*e.g.*, [96, 113]) for elaboration.

2.2 Point particles

For simplicity and practicality, this thesis is, ultimately, concerned with the classical behaviour of *point particles*: particles of zero spatial extent. (*See* Section A.3.15.) Following the guidelines of Section 2.1, we do not concern ourselves with the question of whether any physical particles *are*, in fact, pointlike, but rather simply note that concentrating our attention to point particles both leads to considerable simplification of the equations of motion, and is found to be in practice a most useful approximation.

It should be noted, in passing, that it is *not* correct to use the adjectives *pointlike* and *structureless* interchangeably, and such practice should be eradicated. For example, one may quite validly consider a classical model of an electric dipole as two electric charges on the ends of a stick; if this is shrunk to infinitesimal size, then it is *pointlike*, but arguably not *structureless*. Mathematically, equating these two adjectives correponds to deeming that any non-trivial function of zero extent be a *delta function*; but one may also have *derivatives* of delta functions (as with the point dipole example above) that are of zero extent.

However, the converse generalisation is arguably a reasonable one: any object that is *not* pointlike must have some sort of “structure” keeping it a finite size, even if, in some cases, this structure might be rather featureless.

2.3 Newtonian mechanics

The formulation of mechanics that is arguably the simplest to understand is that originally set out by Newton, summarised at the head of this chapter. In Newton’s way of reckoning, each body possesses a “quantity of matter”, which we now refer to as its mass, m, and a “quantity of motion” [158],

$$\boldsymbol{p} \equiv m\boldsymbol{v} \tag{2.1}$$

(where $\boldsymbol{v}$ is the velocity of the body), which does not change unless some sensible *force* is applied to the body. This definition of force would be circular, except for the fact that Newton *prescribed* the forces that should apply to objects, as Laws of Nature. Of course, Newton himself only laid down a law for gravitation, not one for electrodynamics, nor the laws by which the “particles of bodies cohere in regular figures”; but from his comments above we see that he believed that ultimately *all* of physics could be described in terms of a few universal forces between elementary types of object.

We shall now review various elementary aspects of the Newtonian formulation of mechanics. In some cases we shall make definitions that do not accord with popular practice; these definitions are those the author feels are most appropriate in the task of removing some of the confusion that often surrounds the concepts involved.

2.3.1 Mechanical momentum

Today, we refer to Newton's "quantity of motion", $\boldsymbol{p}$, by the term *mechanical momentum*. In what may at first appear to be pedantry, *the author will never delete the adjective "mechanical" from the term "mechanical momentum"*, in this thesis; and, furthermore, *the author will always use the symbol* $\boldsymbol{p}$ *for the mechanical momentum, and will not use this symbol for any other quantity*.

The reason for the author's insistence on this point will be manifestly clear by the end of Chapter 4.

If one trades-in Galilean mechanics in favour of *Einsteinian* mechanics (to be discussed in more detail shortly), the definition of the mechanical momentum is simply upgraded:

$$\boldsymbol{p} \equiv m\gamma\boldsymbol{v},$$

where every appearance of a velocity $\boldsymbol{v}$ implies the definition of a corresponding *gamma factor*,

$$\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{v}^2}},$$

and where our choice of units is described in Section A.3.4.

2.3.2 Mechanical energy

Of a supplementary but useful rôle in Newtonian physics is the *mechanical energy* of a particle, which, in Galilean mechanics, is given by

$$W \equiv \frac{1}{2}m\boldsymbol{v}^2. \tag{2.2}$$

Again, the author shall *always* refer to the quantity (2.2) using the adjective *mechanical*, and shall *always* use the symbol W.

In upgrading to Einsteinian mechanics, one finds that the mechanical energy is in fact given by the expression

$$W \equiv m\gamma. \tag{2.3}$$

Expanding (2.3) as a power series in $\boldsymbol{v}$, one finds

$$W = m + \frac{1}{2}m\boldsymbol{v}^2 + \mathrm{O}(\boldsymbol{v}^4). \tag{2.4}$$

The first term in (2.4), m, is referred to as the *mechanical rest-energy*. The second term shows that the Galilean result (2.2) is the first correction to the mechanical energy, *i.e.*, the lowest-order term that has a velocity dependence. The full mechanical energy, minus the rest-contribution, is often referred to as the *kinetic energy*,

$$W_k \equiv m(\gamma - 1).$$

The mechanical energy and mechanical momentum in fact constitute a *four-vector* in relativistic physics:

$$p \equiv mU = (m\gamma, m\gamma\boldsymbol{v}),$$

where we are applying the notation and conventions of Section A.8.

2.3.3 Forces

As described above, the fundamental concept in Newtonian mechanics is the *force*. For a body of mass m, the force due to some outside agent is defined to be the *time rate of change of the mechanical momentum*, when the force in question is the only force applied to the body:

$$\boldsymbol{F} \equiv d_t\boldsymbol{p}. \tag{2.5}$$

To upgrade from Galilean to Einsteinian mechanics, the only changes required in (2.5) are that the lab-time derivative is converted into the proper-time derivative, and the left-hand side needs to be given a distinguishing label (*see* Section A.8):

$$f \equiv d_\tau p. \tag{2.6}$$

Note that (2.5) (or (2.6)) is the *only* definition of force admitted in this thesis.

It is an assumption of Newtonian mechanics that *forces are superposable*: the net force on a body is the vectorial sum of the various forces applicable to it. (This applies to Galilean and Einsteinian mechanics equally.) This of course implies that the mechanical momentum of a body is a quantity that is "collected" by the body in a *linear fashion*.

2.3.4 Free particles

In Newtonian mechanics, a particle is "free" if it is not under the influence of any known force. The Newtonian description of a free particle is thus simply that the force on it vanishes:

$$\boldsymbol{F} = \boldsymbol{0},$$

so its mechanical momentum is a constant:

$$d_t \boldsymbol{p} = \boldsymbol{0}. \tag{2.7}$$

Of course, this definition is quite circular, albeit intuitively understandable: one only knows that a particle *is* free because it travels in a straight line at constant speed.

2.3.5 The electromagnetic field

In classical physics, one takes the electromagnetic field to be a classical, continuous c-number field.

It does not really make any sense to talk about the electromagnetic field in terms of Galilean kinematics: since the photon is massless, there is no "non-relativistic limit" in which the field transforms even approximately correctly via the Galilean transformations.

In Einsteinian mechanics, the electromagnetic field satisfies *Maxwell's equations*:

$$\nabla\cdot\boldsymbol{B} \equiv 0, \tag{2.8}$$

$$\nabla\times\boldsymbol{E} + \partial_t\boldsymbol{B} \equiv \boldsymbol{0}, \tag{2.9}$$

$$\nabla\cdot\boldsymbol{E} \equiv \rho, \tag{2.10}$$

$$\nabla\times\boldsymbol{B} - \partial_t\boldsymbol{E} \equiv \boldsymbol{J}. \tag{2.11}$$

The homogeneous Maxwell equations (2.8) and (2.9) tell one that the *six* components of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ may in fact be completely specified in terms of *four* quantities $A^\mu \equiv (\varphi, \boldsymbol{A})$:

$$\begin{aligned}\boldsymbol{E} &\equiv -\nabla\phi - \partial_t\boldsymbol{A},\\ \boldsymbol{B} &\equiv \nabla\times\boldsymbol{A}.\end{aligned} \tag{2.12}$$

However, the A^μ are not directly manifested physically. They are also arbitrary up to a *gauge transformation*:

$$A_\mu \longrightarrow A_\mu + \partial_\mu\Lambda.$$

The electromagnetic field also possesses *mechanical energy, mechanical momentum and mechanical angular momentum densities*. These are given by

$$W_\rho(\boldsymbol{x}) = \frac{1}{2}\Big\{\boldsymbol{E}^2(\boldsymbol{x}) + \boldsymbol{B}^2(\boldsymbol{x})\Big\}, \tag{2.13}$$

$$\boldsymbol{p}_\rho(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{x})\times\boldsymbol{B}(\boldsymbol{x}), \tag{2.14}$$

$$\boldsymbol{s}_\rho(\boldsymbol{x}) = \boldsymbol{x}\times\boldsymbol{p}_\rho(\boldsymbol{x}). \tag{2.15}$$

The former two densities are a part of the electromagnetic *mechanical stress-energy tensor*, $T^{\mu\nu}$:

$$T^{00} = \frac{1}{2}\Big\{\boldsymbol{E}^2 + \boldsymbol{B}^2\Big\},$$

$$
\begin{aligned}
T^{0i} \equiv T^{i0} &= \boldsymbol{E}\times\boldsymbol{B}, \\
T^{ij} &= \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2)\delta^{ij} - \boldsymbol{E}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{B}.
\end{aligned} \tag{2.16}
$$

The mechanical stress-energy tensor $T^{\mu\nu}$ is *always symmetric*—not just for the electromagnetic field, but in complete generality. We shall *always* use the symbol $T^{\mu\nu}$ for the mechanical stress-energy tensor, and not for any other quantity; and we shall *always* use the adjective "mechanical".

The mechanical angular momentum density (2.15) is likewise contained in the *mechanical angular momentum density rank-3 tensor*:

$$
M^{\alpha\beta\gamma} \equiv T^{\alpha[\beta}x^{\gamma]}.
$$

We shall *always* use the symbol $M^{\alpha\beta\gamma}$ for the mechanical angular momentum tensor, and not for any other quantity; and shall *always* use the adjective "mechanical".

2.3.6 Electrically charged particles

If a particle possesses an *electric charge* q, then the corresponding Law of Nature for its Newtonian force is the *Lorentz force law*:

$$
\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{2.17}
$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic field at the position of the particle respectively.

The *power into* the electric charge,

$$
P \equiv d_t W,
$$

is, for an electric charge, simply given by

$$
P = q(\boldsymbol{v}\cdot\boldsymbol{E}), \tag{2.18}
$$

which simply corresponds to the change in mechanical energy implied by the Lorentz force (2.17); there is *no* power into a stationary electric charge.

In Einsteinian mechanics, the Lorentz force (2.17) is actually still correct, if $\boldsymbol{F}$ is taken to be $d_t\boldsymbol{p}$; in manifestly covariant terms, the result is

$$f = qF\cdot U.$$

The vanishing of the power (the zero-component of f) for $\boldsymbol{v} = \boldsymbol{0}$ then tells us that the *mass of an electric charge is a constant of the motion:*

$$\dot{m} \equiv f\cdot U = 0. \tag{2.19}$$

The result (2.19) may seem trivial, but it is *not* the case for an arbitrary *dipole* moment (*see* Chapter 4), which *may* gain mechanical rest-energy, in the general case.

2.3.7 Conservation laws

It is a postulate of Newtonian physics that, for a closed system, mechanical energy, mechanical momentum and mechanical angular momentum are *always conserved*.

It was primarily to ensure that these conservations laws were satisfied that the electromagnetic field was ascribed the mechanical densities (2.13), (2.14) and (2.15). Thus, in one sense, the satisfaction of the conservation laws was "rigged" after the fact; but one must also note that it is a non-trivial feature that such conservation *can* be achieved by defining mechanical field densities that are *local*, and involve *only the quantities* $\boldsymbol{E}(\boldsymbol{x})$ *and* $\boldsymbol{B}(\boldsymbol{x})$.

2.3.8 Conservation derivation of the Lorentz force

If one examines the situation carefully, one finds that the definitions made above for the Lorentz force law, the Maxwell equations, the mechanical field densities and the conservation laws contain a redundancy: given any three definitions, the fourth actually follows as a consequence.

The choice of which definition is "consequential" is of course arbitrary. In practice, either the mechanical field density or the Lorentz force is generally derived from the other three. Let us illustrate the latter, with a simplified but indicative outline of the argumentation required: If one considers a system of just *two electric charges*, of finite separation, then, when they are both at rest, it is simple to show that the *excess in the mechanical field energy*, compared to the case when the two charges are infinitely separated, is simply the Coulomb-gauge scalar potential of either particle, evaluated at the position of the other particle, and multiplied by this other particle's charge:

$$\Delta W_{\text{field}} = q_1 \varphi_2(\boldsymbol{z}_1) = q_2 \varphi_1(\boldsymbol{z}_2).$$

(*See*, *e.g.*, Jackson [113, Sec. 1.11]; a copy of this proof is supplied in Section C.2.1.) In the general case, the second charge may be replaced by an arbitrary number of generating charges (*see* Section C.2.1); one then has

$$W_{\text{excess}} = q\varphi(\boldsymbol{z}),$$

where we write q for the charge being brought into the midst of the other charges generating $\varphi(\boldsymbol{r})$; $\boldsymbol{z}$ is the position of the charge q. Similarly, when a charge is brought into the vicinity of a *magnetic* field distribution $\boldsymbol{B}(\boldsymbol{r})$, the *excess in the mechanical field momentum* can be shown to be the Coulomb-gauge *vector* potential of the sources generating the $\boldsymbol{B}(\boldsymbol{r})$, evaluated at the position of the charge brought in:

$$\Delta \boldsymbol{p}_{\text{field}} = q\boldsymbol{A}(\boldsymbol{z}).$$

(A proof of this result is supplied in Section C.2.2.) Now, the elementary "conservation of mechanical energy" derivation of the force on an object (*see*, *e.g.*, [96]) shows that the mechanical momentum $\boldsymbol{p}$ of the object must satisfy

$$d_t \boldsymbol{p} + \nabla W_{\text{excess}} = \boldsymbol{0},$$

in order to conserve energy. But this elementary derivation assumes that only the mechanical *energy* of the system is affected by the position and velocity of the object. Its extension to the case where the mechanical *momentum* is also affected is straightforward: one simply finds, for the case of the electric charge above,

$$d_t\Big\{\boldsymbol{p}+q\boldsymbol{A}(\boldsymbol{z})\Big\}+q\nabla\Big\{\varphi(\boldsymbol{z})-\boldsymbol{v}\cdot\boldsymbol{A}(\boldsymbol{z})\Big\}=\boldsymbol{0}, \tag{2.20}$$

in order that the total mechanical energy and mechanical momentum of the system be conserved. By noting the vector identities of Section B.3, one can quickly show that the equation of motion (2.20) yields simply the *Lorentz force law*, (2.17).

The author does not believe he has seen an analysis such as the one above for the case of arbitrarily accelerated particles (since then the arbitrarily violent effects of retardation are not to be ignored), in arbitrarily relativistic motion, but one would expect that the same result would hold, since we know that the Lorentz force *does* work, experimentally, regardless of the particular motion of the source of the electric and magnetic fields experienced. Furthermore, we shall find, shortly, a method of derivation of the Lorentz force quite similar to the one reviewed above, but which is in fact a *local* formulation of the problem, which *can* therefore be assumed to hold for arbitrary motion of the particles involved.

Any derivation of a force equation of motion using such "conservation of mechanical four-momentum" principles must be performed extremely carefully, since one must integrate expressions over all space, and take into account all variations in the motion of the particles involved. One must also, of course, ensure that one deals *solely with the mechanical quantities as defined above*; this seems like a trivial warning, but it has been ignored a surprising number of times. However, if all subtleties are given due respect, this method of derivation is most definitely valid.

2.4 Lagrangian mechanics

An alternative, and most fruitful, formulation of classical mechanics is that due to Lagrange and Hamilton. One postulates a *Lagrangian function*, L, that depends on some number n of generalised coördinates of the system, labelled q_a (where a runs from 1 to n); their first time-derivatives, denoted $\dot{q}_a$; and the time t. One then further postulates that the motion of the system from time t_1 to time t_2 is such that the line integral

$$I \equiv \int_{t_1}^{t_2} L \, dt, \tag{2.21}$$

referred to as the *action*, has a *stationary value* for the actual motion of the system, where the two end-points are fixed. From this principle, it a merely mathematical problem to show [96] that the system evolves according to the *Euler–Lagrange equations*:

$$d_t\Big(\partial_{\dot{q}_a} L\Big) - \partial_{q_a} L = 0. \tag{2.22}$$

Because the end-points of the variation are fixed, the Euler–Lagrange equations are unchanged if a *total time derivative* is added to the Lagrangian:

$$L \longrightarrow L + d_t \mathit{\Gamma}.$$

We thus seen that, in Lagrangian mechanics, the fundamental mathematical quantity is the *Lagrangian function*; equally important is a knowledge of the *appropriate degrees of freedom*, q_a, for the physical system in question.

We shall now describe various definitions and specifications, for the use of the Lagrangian formulation of mechanics in this thesis, as was done in Section 2.3 for the Newtonian formulation.

2.4.1 Generalised velocities

If the degree of freedom q_a is a "generalised coördinate" for the physical system in question, then it is clearly suitable for one to refer to the quantity

$$\dot{q}_a \tag{2.23}$$

as the "generalised velocity" for that degree of freedom. However, no particular symbol will be assigned to this quantity other than the explicit expression (2.23), except where it happens to coïncide with another quantity that does possess a special symbol.

2.4.2 Canonical momentum

It is convenient to refer to the quantity

$$\partial_{\dot{q}_a} L \tag{2.24}$$

appearing in the Euler–Lagrange equation (2.22) by a new name and notation. Unfortunately, due to historical reasons, this quantity has generally been referred to by a symbol already reserved by the author above; and by a term confused with one used by the author above. To avoid any possibility of confusion or error, in this thesis, the author shall *always* refer to the quantity (2.24) as the *canonical momentum conjugate to the coördinate* q_a, or simply a *canonical momentum coördinate*; but under no circumstances will the adjective "canonical" be omitted. He will furthermore assign a *new symbol* to this quantity, with no particular historical precedent to justify his choice,—and indeed no particular justification *at all* for the choice, other than the fact that it is as easy to write and print as the conventional symbol—being simply a spatial reflection of it,—while being unambiguously distinguishable from the quantities already defined in the previous sections; but most importantly being the symbol the author has simply become used to using for the canonical momentum:

$$b_a \equiv \partial_{\dot{q}_a} L. \tag{2.25}$$

Readers who work exclusively in the field of Quantum Mechanics will no doubt protest at the author's choice of symbol for the canonical momentum, running contrary to the popular choice p; but readers who work exclusively in the field of General Relativity will without an equal amount of doubt welcome it with open arms—the symbol p being standard notation for the *mechanical* momentum; and readers who work in both fields are probably simply glad the author realises that a notational conflict exists *at all*;—gladder still that the author will scrupulously avoid any pathetic confusion between p and b in the remaining pages of this thesis.

To summarise, the definition (2.25) states the following, according to the author's notation and nomenclature: the canonical momentum coördinate b_a, conjugate to the generalised coördinate q_a, is given by the derivative of the Lagrangian L with respect to the generalised velocity $\dot{q}_a$ of that coördinate.

2.4.3 The Hamiltonian function

The Euler–Lagrange equations of motion (2.22) above are obtained from the Lagrangian L, which is a function of the generalised coördinates, the generalised velocities, and the time. We may alternatively seek a description of the system in terms of the generalised coördinates, the *canonical momenta*, and the time. It may be shown (*see*, *e.g.*, [96]) that the equations of motion of the system *can* be so framed, and are encapsulated in the *Hamiltonian function*,

$$H(q_a, b_a, t) \equiv \dot{q}_a b_a - L(q_a, \dot{q}_a, t), \tag{2.26}$$

where we are employing the Einstein summation convention of Section A.3.8. Starting from the Euler–Lagrange equations (2.22) one then finds that the generalised coördinates and canonical momenta evolve according to

$$\dot{q}_a \;=\; \partial_{b_a} H, \tag{2.27}$$

$$-\dot{b}_a = \partial_{q_a} H, \tag{2.28}$$

and the partial time derivatives of the Lagrangian and Hamiltonian functions are themselves related by

$$-\partial_t L = \partial_t H.$$

When one examines the Hamiltonian formalism from the point of view of relativistic mechanics, one finds that the Hamiltonian function is in fact the *zero component of a four-vector*, the spatial parts of which are the canonical momentum coördinates conjugate to the spatial translational degrees of freedom of the system as a whole. This four-vector is *always* referred to as the *canonical four-momentum* in this thesis, and is *always* denoted

$$b^\mu \equiv (b^0, \boldsymbol{b}) \equiv (H, b^x, b^y, b^z). \tag{2.29}$$

In quantum mechanics, one computes *eigenvalues* of the Hamiltonian operator H. These are referred to as *canonical energy eigenvalues* in this thesis, or simply *canonical energies*; the adjective "canonical" is *never* omitted. They are *always* denoted by the symbol

$$E,$$

and this (scalar) symbol is *never* used for any other quantity.

2.4.4 Free particles

The case of a free particle is not as trivial in Lagrangian mechanics as it is in Newtonian mechanics: we must determine the *degrees of freedom* applicable to a free particle, as well as a suitable *Lagrangian function*. For the former consideration we take the three components of the spatial position of the particle,

$$\boldsymbol{z}, \tag{2.30}$$

to be the appropriate degrees of freedom q_a. For the free-particle Lagrangian, one simply postulates the function

$$L = \frac{1}{2}m(d_t\boldsymbol{z})^2 \equiv \frac{1}{2}m\boldsymbol{v}^2. \tag{2.31}$$

The Euler–Lagrange equations (2.22) for the three degrees of freedom $\boldsymbol{z}$ then yield

$$d_t(m\boldsymbol{v}) = \boldsymbol{0}. \tag{2.32}$$

To provide a more explicit (albeit in this case quite trivial) comparison with the result of *Newtonian* mechanics, we can alternatively write the result (2.32) in terms of the *mechanical momentum* $\boldsymbol{p}$—keeping uppermost in our minds that mechanical quantities are *not* actually native to Lagrangian mechanics—to yield

$$d_t\boldsymbol{p} = \boldsymbol{0},$$

which is of course identical to (2.7).

We can also analyse the free particle in the Hamiltonian formalism. From the Lagrangian (2.31), we find that the *canonical momentum components* $\boldsymbol{b}$ conjugate to the degrees of freedom $\boldsymbol{z}$ are given by

$$b_i \equiv \partial_{v_i} L = mv_i;$$

in other words,

$$\boldsymbol{b} = m\boldsymbol{v}. \tag{2.33}$$

Thus, for a free particle, the canonical momentum and mechanical momentum are equal. But it should be most carefully noted that, out of *all* the systems considered in this thesis, *this equality of canonical and mechanical momentum only holds true for the case a free particle*; it is in large part due to this equivalence in one *special* case that these two quantities have, historically, been *erroneously* equated in general situations.

Let us return to the expression (2.33) for the free-particle canonical momentum. Using this in (2.26), we find

$$H = \boldsymbol{v}\cdot(m\boldsymbol{v}) - \frac{1}{2}m\boldsymbol{v}^2;$$

hence, the *value* of the Hamiltonian is simply given by

$$H = \frac{1}{2}m\boldsymbol{v}^2.$$

However, we must now eliminate the generalised velocity coördinates $\boldsymbol{v}$ in favour of the canonical momentum coördinates $\boldsymbol{b}$, since H must be expressed as a function of the q_a and b_a (and t) only, not the $\dot{q}_a$. We can perform this elimination by using the relation (2.33), yielding

$$H(\boldsymbol{z}, \boldsymbol{b}, t) = \frac{\boldsymbol{b}^2}{2m}.$$

We can now use the Hamiltonian H as a *starting point*, pretending that we had not seen its method of derivation above. The first set of Hamilton's equations (2.27) yield

$$\boldsymbol{v} = \frac{\boldsymbol{b}}{m}, \tag{2.34}$$

which seems trivial, but which is (in more general situations) a vitally important piece of information. The second set of Hamilton's equations (2.28) yields

$$d_t\boldsymbol{b} = \boldsymbol{0}, \tag{2.35}$$

or, on using (2.34),

$$d_t(m\boldsymbol{v}) = \boldsymbol{0},$$

again the elementary Newtonian result.

2.4.5 The electromagnetic field

We must now consider the question of obtaining a description of the classical electromagnetic field in Lagrangian terms. This is not, *a priori*, a trivial task. Firstly, we need to select appropriate *degrees of freedom* for the electromagnetic field. We then need to postulate a suitable *Lagrangian function* that reproduces the Maxwell equations.

Fortunately, this problem has been around for a long time, and a suitable formulation found: one treats the four components of the *four-potential* A^μ as the degrees of freedom for the field, at each and every spacetime point x. It is straightforward to verify (*see*, *e.g.*, [113, Sec. 12.8], or [96, Chap. 12]) that an action principle for a *continuous field* of degrees of freedom $\phi(x)$, over some space x, yields Euler–Lagrange equations of motion

$$\partial_\alpha\Big(\partial_{\partial_\alpha \phi}\mathcal{L}\Big) - \partial_\phi \mathcal{L} = 0, \tag{2.36}$$

where $\mathcal{L}$ is a Lagrangian density in the space x.

For the electromagnetic field, a suitable Lagrangian density is given by [113, Sec. 12.8]

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - J^\alpha A_\alpha, \tag{2.37}$$

where $J^\alpha(x)$ is the electromagnetic source current density,

$$J(x) \equiv (\rho(x), \boldsymbol{J}(x)),$$

and where the notation $F_{\alpha\beta}$ in (2.37) is, in this case, merely a shorthand way of writing the derivatives of the four-potential components:

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{2.38}$$

Applying the Euler–Lagrange equations (2.36) to the Lagrangian (2.37) with the identification (2.38), one finds

$$\partial \cdot F = J,$$

the covariant expression of the inhomogeneous Maxwell equations. (The *homogeneous* Maxwell equations are axiomatic if we assume the four-potential $A(x)$ to be fundamental.)

The electromagnetic field also possesses *canonical energy, canonical momentum and canonical angular momentum*. These are most simply written in terms of the *canonical stress-energy tensor*, $\Theta^{\mu\nu}$, which, for a set of continuous field $\phi_a(x)$, is defined as [113, Sec. 12.10]

$$\Theta^{\alpha\beta} \equiv \partial^{\beta}\phi_a \cdot \partial_{\partial_\alpha \phi_a}\mathcal{L} - g^{\alpha\beta}\mathcal{L}. \tag{2.39}$$

(We use a different notation to that of Jackson, whose use of $T^{\mu\nu}$ for two different quantities is confusing.) The canonical stress-energy tensor $\Theta^{\alpha\beta}$ is the covariant field generalisation of the definition (2.26) of the Hamiltonian function H (which is, as noted, simply the *canonical energy* of the system). For the electromagnetic field, one finds [113, Sec. 12.10]

$$\Theta^{\alpha\beta} = -F^{\alpha}{}_{\mu}\partial^{\beta}A^{\mu} + \frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} + g^{\alpha\beta}J^{\mu}A_{\mu}. \tag{2.40}$$

One can similarly define the *canonical angular momentum rank-3 tensor*:

$$\Xi^{\alpha\beta\gamma} \equiv \Theta^{\alpha[\beta}x^{\gamma]}. \tag{2.41}$$

We shall *always* use the adjective "canonical" when referring to the tensors (2.39) and (2.41), and will *always* use the symbols $\Theta^{\mu\nu}$ and $\Xi^{\alpha\beta\gamma}$.

It should be noted that the canonical stress-energy tensor (2.39) (or (2.40)) and the canonical angular momentum tensor (2.41) are *not symmetric*. They also do not, in general, have any direct connection with the *mechanical* stress-energy and angular momentum tensors defined in Section 2.3.5, although certain terms are contained in both (*see*, *e.g.*, [113, Sec. 12.10(b)]).

On the other hand, the canonical stress-energy tensor (2.39) possesses other qualities, that (if properly used) can make it even more powerful than its mechanical counterpart: it can be obtained for arbitrary fields, not just the electromagnetic;—its definition is set, rather than expressions postulated for each physical application; it automatically proves the *conservation of* *canonical energy, canonical momentum and canonical angular momentum* for a closed system; and it has a much closer connection with *group theory*, which is formulated completely in canonical—*not* mechanical—terms.

2.4.6 Electrically charged particles

We now consider the problem of obtaining the motion of a charged particle from a Lagrangian viewpoint. The source term $-J^\mu A_\mu$ of (2.37) suggests that we should try the Lagrangian

$$L = \frac{1}{2} m\boldsymbol{v}^2 - q(\varphi - \boldsymbol{v}\cdot\boldsymbol{A}) \tag{2.42}$$

for the nonrelativistic charged particle. Computing first the canonical momentum coördinates $\boldsymbol{b}$ conjugate to the position variables $\boldsymbol{z}$, we find

$$b_i \equiv \partial_{v_i} L = m\boldsymbol{v} + q\boldsymbol{A}. \tag{2.43}$$

Now, since the nonrelativistic quantity $m\boldsymbol{v}$ is simply the *mechanical* momentum, we find the most important relation

$$\boldsymbol{b} = \boldsymbol{p} + q\boldsymbol{A}, \tag{2.44}$$

or, alternatively,

$$\boldsymbol{p} = \boldsymbol{b} - q\boldsymbol{A}. \tag{2.45}$$

The result (2.44) or (2.45) is referred to as the *principle of minimal coupling*. It is clear why one most definitely needs to have distinct and unambiguous notation and nomenclature for the mechanical and canonical momenta: even for the simplest electrodynamical case of an *electric charge*, these two quantities are not identical.

Let us now turn immediately to the Euler–Lagrange equations (2.22). For the Lagrangian (2.42), using the result (2.44), we find

$$d_t(\boldsymbol{p} + q\boldsymbol{A}) + q\nabla(\varphi - \boldsymbol{v}\cdot\boldsymbol{A}) = \boldsymbol{0}.$$

Collecting together the terms involving q onto the right-hand side, using the *convective derivative* of the field $\boldsymbol{A}$,

$$d_t\boldsymbol{A} \equiv \partial_t\boldsymbol{A} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A},$$

and employing the vector identities of Section B.3, we thus find [96, Sec. 1–5]

$$d_t\boldsymbol{p} = -q\nabla\varphi - q\partial_t\boldsymbol{A} + q\boldsymbol{v}\times(\nabla\times\boldsymbol{A});$$

noting the identities (2.12), we thus find

$$d_t\boldsymbol{p} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}), \tag{2.46}$$

the Lorentz force expression (2.17).

The derivation leading to (2.46) may seem familiar to the reader. It should: it in fact gives identical expressions to the *mechanical conservation* method of derivation of the Lorentz force law outlined in Section 2.3.8. We can now begin to see the connection between the canonical and mechanical quantities more clearly—but it is most important to get the relationships absolutely correct: the *canonical* momentum of a charged particle includes both the *mechanical* momentum of the charge, plus the *mechanical momentum excess* contained in the electromagnetic field due to the electric charge's vicinity to the other charges in the Universe generating the "external" fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

Note carefully the concepts involved here: One can derive the Lorentz force *either* by insisting on conservation of the total mechanical momentum of the system—particle *plus* field;—or one can use the Lagrangian framework to obtain the equation of motion for the *canonical* momentum expression, *which itself includes the mechanical field momentum excess*. These methods of derivation are mutually exclusive: one *does not and cannot* add the "hidden field momentum" to the Lorentz force law itself: *it has already been taken account of*. This point may seem obvious; but a careful and unambiguous recognition of the subtle issues involved here will be *vital* for the considerations of Chapter 4.

Let us return to the electric charge Lagrangian (2.42). We can obtain a *Hamiltonian* description of the system by again using the result (2.26):

$$\begin{aligned} H &\equiv \boldsymbol{v}\cdot\boldsymbol{b} - L \\ &= \boldsymbol{v}\cdot(m\boldsymbol{v} + q\boldsymbol{A}) - \left\{\frac{1}{2}m\boldsymbol{v}^2 - q(\varphi - \boldsymbol{v}\cdot\boldsymbol{A})\right\} \\ &= \frac{1}{2}m\boldsymbol{v}^2 + q\varphi. \end{aligned}$$

We again must convert the functional dependence of this result, from being in terms of the generalised velocity components $\boldsymbol{v}$, to being in terms of the canonical momentum coördinates $\boldsymbol{b}$. We can use (2.43) to effect this substitution, yielding

$$H(\boldsymbol{z}, \boldsymbol{b}, t) = \frac{(\boldsymbol{b} - q\boldsymbol{A})^2}{2m} + q\varphi. \tag{2.47}$$

It may be seen why the "minimal coupling" result (2.44) was a guiding light in the early days of electrodynamics: not only does one need to add a scalar potential $q\varphi$ to the free-particle Hamiltonian, one must additionally *modify* the factor $\boldsymbol{b}^2/2m$ into $(\boldsymbol{b} - q\boldsymbol{A})^2/2m$. Of course, this term is *still* simply the nonrelativistic kinetic energy of the particle:

$$\frac{(\boldsymbol{b} - q\boldsymbol{A})^2}{2m} = \frac{\boldsymbol{p}^2}{2m} \equiv \frac{1}{2}m\boldsymbol{v}^2;$$

it is simply that the *functional dependence* of this term on the canonical momentum *changes*, because the canonical momentum itself is *no longer* the mechanical momentum.

The situation may be clarified even further by considering the problem when generalised to Einsteinian mechanics. The form of the Hamiltonian (2.47), which we can rewrite

$$H - q\varphi = \frac{(\boldsymbol{b} - q\boldsymbol{A})^2}{2m},$$

suggests that perhaps the fully relativistic expression is

$$(H - q\varphi)^2 = m^2 + (\boldsymbol{b} - q\boldsymbol{A})^2. \tag{2.48}$$

In fact, this is indeed the case [113, Eq. (12.14)]. We may alternatively write (2.48) in the form

$$H = \sqrt{m^2 + (\boldsymbol{b} - q\boldsymbol{A})^2} + q\varphi,$$

which most clearly shows how the nonrelativistic result (2.47) is obtained as the first-order expansion of the square-root (ignoring the functionally trivial—but relativistically important—zeroth-order term m).

The manifest covariance of the result (2.48) may be exhibited explicitly by recalling that the Hamiltonian is in fact the *canonical energy* of the system—the zero component b^0 of the canonical four-momentum b^α. The relation (2.48) then simply states that

$$p^2 = (b - qA)^2 = m^2, \tag{2.49}$$

since φ is the zero-component A^0 of A^μ. We can now recognise the *fully relativistic* formulation of the "minimal coupling" result:

$$p = b - qA; \tag{2.50}$$

or, alternatively,

$$b = p + qA. \tag{2.51}$$

It must again be stressed that the identities

$$p \equiv mU$$

and (as a consequence)

$$p^2 \equiv m^2$$

are sacrosanct; hence, once one has recognised the result (2.50), the covariant expression (2.49) for the Hamiltonian $H \equiv b^0$ follows immediately. It should also be carefully noted again that *the canonical momentum has no relationship to the velocity of the particle*: in particular, we have

$$b^2 \neq m^2,$$

in the general case.

Finally, it must be noted that *all* of the results above linking the mechanical and canonical momenta apply *only to the case of an electric charge*: when we add more complicated moments, in Chapter 4, *we shall have to start this procedure all over again*. In particular, the "minimal coupling" result (2.50) is *not* universal. (However, we shall see that it may in fact be generalised.)

2.4.7 Conservation laws

We now briefly return to a subject touched on above, that will not be of any practical importance in this thesis, but which is most powerful for a number of applications: the ability to extract *conservation theorems* from the Lagrangian or Hamiltonian description of a physical system.

For the case of discrete degrees of freedom q_a, the Euler–Lagrange equations (2.22)—or equivalently Hamilton's equations (2.28)—show that, if a generalised coördinate q_a does not explicitly appear in the Lagrangian or Hamiltonian, then the corresponding conjugate canonical momentum b_a is *constant in time*. This finding is *not* restricted to the translational degrees of freedom of a system: it applies to any generalised coördinate of any suitably describable system.

Similarly, for the case of *continuous field* degrees of freedom $\phi(x)$, one can extract powerful conservation laws. If the Lagrangian density $\mathcal{L}$ is *invariant under a continuous one-parameter set of transformations*, then it can be shown (*see*, *e.g.*, [145, Sec. 2.4]) that the Euler–Lagrange equations (2.36) yield a corresponding *conserved quantity*; this is known as *Noether's theorem*.

We shall not elaborate further on these powerful physical and analytical tools, but will merely remind the reader that *such considerations always* *apply to canonical quantities*, being as they are derived from the Lagrangian function L or $\mathcal{L}$.

2.5 The canonical–mechanical challenge

The author challenges the reader to pick up an arbitrarily chosen textbook or research paper from their bookshelf, leaf through the chosen volume, and count how many times the concepts of *canonical quantities* and *mechanical quantities* are either confused, mistaken for each other, written in ambiguous or oft-changed notation, or simply not recognised at all.

The reasons for the author's apparent pedantry, in the preceding sections, will then be clear.

As a rule, the *best* authors generally define completely separate symbols for mechanical and canonical quantities—although the choices made are not at the present time standardised; *good* authors generally only mix their notation: the concepts are clearly understood and enunciated; but less fortunate authors mix the two types of quantity together with gay abandon, generally leading to completely meaningless and useless conclusions.

Reader beware!

2.6 Relativistic mechanics

In this section, we consider the various subtleties introduced by the use of the mechanics of Einstein's Special theory of Relativity [75]. Again, some results are simply listed here as an introduction to the author's notation; other, more subtle aspects are discussed in somewhat more detail.

2.6.1 Mass

The *mass* m of a system is a Lorentz scalar, and is defined as the *mechanical energy of the system in the rest frame of the system.*

For an elementary particle, the mass m is a constant.

We do *not* employ the terms "rest mass", "moving mass", "longitudinal mass", *etc.*, in this thesis.

2.6.2 Four-position

The *four-position* of a classical point particle is denoted

$$z \equiv (t, \boldsymbol{z}).$$

The four-position is a *Lorentz-covariant quantity*, or simply a *covariant quantity*, in the nomenclature of Section A.8.

2.6.3 Three-velocity

The three-velocity $\boldsymbol{v}$ of a classical particle is defined as

$$\boldsymbol{v}(t) \equiv d_t \boldsymbol{z}(t).$$

The three-velocity is a *non-covariant quantity*, in the nomenclature of Section A.8.

2.6.4 Proper time

A most important concept in relativistic mechanics is that of the *proper time*, τ. It is the time cumulated in the *co-accelerated coördinate system* (or *CACS*) of the particle.

At any particular instant in time, one may set up a *momentarily comoving Lorentz frame* (or *MCLF*) for the particle, which is, as its name suggests, a global Lorentz frame in which the particle is momentarily at rest.

Although the CACS and MCLF have subtle differences in their behaviour, the differential of the proper-time *may* in fact be computed around the rest-instant in the MCLF:

$$d\tau = dt|_{\text{MCLF}}.$$

Since the MCLF is a Lorentz frame, we can boost the right-hand side of this result to an arbitrary, "lab" frame, in which the particle moves with three-velocity $\boldsymbol{v}$:

$$d\tau \equiv \frac{dt}{\gamma} \equiv dt\sqrt{1-\boldsymbol{v}^2}. \tag{2.52}$$

2.6.5 Four-velocity

The *four-velocity* of a classical point particle is defined as

$$U(\tau) \equiv d_\tau z(\tau).$$

From (2.52) it is clear that the components of U are given by

$$U = (\gamma, \gamma\boldsymbol{v}),$$

and hence

$$U^2 \equiv 1. \tag{2.53}$$

2.6.6 Mechanical four-momentum

The *mechanical four-momentum* of a classical particle is *always* defined as

$$p \equiv mU.$$

Thus,

$$p^2 = m^2$$

always. However, in the general case of systems of *arbitrary mass* (*i.e.*, arbitrary mechanical rest-energy), the mass m may itself be a function of time, rather than simply a constant. Except where otherwise noted, we shall *not* assume the mass of a system to be a constant.

2.6.7 Unit four-spin

It is often necessary to consider an "internal" property of a classical point particle, that may be fully described in terms of *a spatial direction in the particle's rest frame*. For such general purposes we define the unit vector

$$\boldsymbol{\sigma} \tag{2.54}$$

in this frame. We shall generally refer to this three-vector (2.54) as the *unit three-spin*, since for the physically important case of a spin-half particle it shall represent the (expectation value of the) spin of the particle, normalised to unity; in the general case, however, this vector may be used for arbitrary purposes, not just to represent spin.

To provide a relativistic generalisation of this three-vector, we define the *unit four-spin*, $\mathit{\Sigma}$, so that, in the instantaneous rest frame of the particle, this four-vector has components

$$\mathit{\Sigma}|_{\text{MCLF}} = (0, \boldsymbol{\sigma}). \tag{2.55}$$

Subtleties involved in taking time-derivatives of the four-spin are discussed in the following two sections.

2.6.8 The Thomas precession

From the discussion of the previous section, we can obtain the explicit components of the unit four-spin $\mathit{\Sigma}$ in *any* Lorentz frame, by means of the Lorentz transformation (G.1) of the rest-frame definition (2.55):

$$\begin{aligned} \mathit{\Sigma}^0 &= \gamma(\boldsymbol{v}\cdot\boldsymbol{\sigma}), \\ \boldsymbol{\Sigma} &= \boldsymbol{\sigma} + \frac{\gamma^2}{\gamma+1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}. \end{aligned} \tag{2.56}$$

There is, however, a subtlety. Since the only requirements of the MCLF and lab frames are that they give the appropriate *velocity* of the particle (**0** and

$\boldsymbol{v}$ respectively), there remains an ambiguity of the *spatial orientation* of the two frames. For the four-velocity U, this ambiguity is irrelevant, because U only depends on the three-velocity $\boldsymbol{v}$. But for the unit four-spin Σ, this problem is of course *maximal*, because $\boldsymbol{\sigma}$ itself has been introduced for the very purpose of *specifying* a spatial direction in the rest frame.

From our writing down of the definition (2.56), however, we *have* at least specified a definite relationship between the MCLF and the lab frame at some particular time: the directions of the spatial axes of the MCLF are coïncident with those of the chosen lab frame. The problem arises in the infinitesimal time period following this instant. From (2.56), it would seem that we could describe the time-evolution of the four-spin of the particle most simply through the three-vector

$$\boldsymbol{\sigma}(t), \tag{2.57}$$

which we would reasonably believe to be "the rest-frame" three-spin of the particle. This is, indeed, a mathematically valid way to proceed, but our interpretation must be tempered somewhat. In using (2.57) in (2.56), we are effectively performing *a boost from the lab frame to the particle's rest frame for each instant of time*. This means that, as far as the particle is concerned, we are, at each instant of time, referring its three-spin back to the *lab* frame, by a boost of $-\boldsymbol{v}$; we then evolve the three-spin by an amount $\dot{\boldsymbol{\sigma}}\,dt$, and then we *boost back to the particle's new rest frame*—which is of course slightly boosted from its previous frame because the particle is accelerated; hence our "boost back" is by the velocity $\boldsymbol{v} + \dot{\boldsymbol{v}}\,dt$.

Now, these operations can be carried out explicitly, using the Lorentz transformation (G.1), and taking due note of the time-dilation factor relating the time coördinates in the two frames. One might *expect* that this whole complicated analysis would simply be equivalent to boosting the particle by velocity $\dot{\boldsymbol{v}}\,dt$ in the rest frame of the particle, without the added steps of a boost to the lab frame and then the boost back. *However, this is not* *the case.* The reason, pointed out by Thomas in 1926 [213, 214], is that *pure Lorentz boosts do not form a group*: indeed, two successive Lorentz boosts are, in the general case, equivalent to one overall Lorentz boost, *plus a rotation.* In terms of the description given above, we may understand the situation as follows: the naïve expectation assumes that the Lorentz boost from the lab frame to the rest frame may be trivially "commutated" through the incremental boost of the rest frame due to its acceleration, "cancelling off" the return boost back to the lab frame. But this is not so: the boosts do not commute; indeed, their commutator is just the rotation corresponding to the "Thomas precession", as it has become called.

If one performs the algebra explicitly (*see*, *e.g.*, Jackson [113, Sec. 11.8]), one finds that

$$\dot{\boldsymbol{\sigma}}|_{\mathrm{lab}} = \frac{1}{\gamma}\dot{\boldsymbol{\sigma}}|_{\mathrm{MCLF}} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\sigma}\times(\boldsymbol{v}\times\dot{\boldsymbol{v}}), \tag{2.58}$$

where the factor of $1/\gamma$ in the first term of the right-hand side is simply the time-dilation relation between the two time coördinates; the second term is the Thomas precession. (We shall return to a covariant derivation of (2.58) in the following section.)

Note that the Thomas precession is a *purely kinematical effect*: it arises *automatically* through the kinematics of the Lorentz group; it does not require any dynamical "force" or "Hamiltonian" in order to bring it about.

It may be wondered how the three-spin of an accelerated particle can be seen to be precessing with *different rates* in two different Lorentz frames, over and above the time-dilation effect: after all, aren't these the simple Lorentz frames of Special Relativity? The answer is that even the description in the MCLF is only valid at *one instant* in time; to obtain even the instantaneous time evolution in *this* frame, we must still employ the Lorentz transformation by $\dot{\boldsymbol{v}}\,dt$. In reality, the physically meaningful rate of precession is actually that measured in the *CACS* of the particle, *not* the MCLF. The fact that the required boost for the MCLF is by a quantity of first order in dt (with no finite $\boldsymbol{v}$ added) means that the measured precession rate in the MCLF is actually identical to that in the CACS, at that instant. But the MCLF of course gets "left behind" by the particle. One thus finds that the CACS in fact *rotates* relative to the various fixed Lorentz frames, if the direction of the particle's velocity is changing.

If the reader finds the Thomas precession completely counter-intuitive, and mind-boggling, the reader is not alone, and may be perversely gratified to find that it also slipped the author's attention on one occasion during the work of this thesis (to be described in Section 3.3.3); but even those for whom the counter-intuitive was second nature were sometimes bamboozled by the concept. We quote from Pais [162]:

> Einstein also pointed out that the transformations
>
> $$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx/c^2), \end{aligned}$$
>
> form a group, 'wie dies sein muss,' as it should be (he did not expand on this cryptic statement): two successive transformations with velocities v_1, v_2 in the same direction result in a new transformation of this form with a velocity v given by
>
> $$v = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$
>
> Twenty years later, Einstein heard something about the Lorentz group that greatly surprised him. It happened while he was in Leiden. In October 1925 George Eugene Uhlenbeck and Samuel Goudsmit had discovered the spin of the electron and thereby explained the occurrence of the alkali doublets, but for a brief

> period it appeared that the magnitude of the doublet splitting did not come out correctly. Then Llewellyn Thomas supplied the missing factor, 2, now known as the Thomas factor. Uhlenbeck told me that he did not understand a word of Thomas's work when it first came out. 'I remember that, when I first heard about it, it seemed unbelievable that a relativistic effect could give a factor of 2 instead of something of order $v/c \ldots$ Even the cognoscenti of the relativity theory (Einstein included!) were quite surprised.' At the heart of the Thomas precession lies the fact that a Lorentz transformation with velocity $\boldsymbol{v}_1$ followed by a second one with velocity $\boldsymbol{v}_2$ in a different direction does not lead to the same inertial frame as one single Lorentz transformation with the velocity $\boldsymbol{v}_1 + \boldsymbol{v}_2$. (It took Pauli a few weeks before he grasped Thomas's point.)

What chance, then, is there for us mere mortals?

2.6.9 Partial and covariant derivatives

We now turn to another subtle but important aspect of relativistic mechanics, not unrelated to the considerations of the previous section, that requires clarification before we may proceed to obtain equations of motion in the following chapters. The problem arises when we try to compute *proper-time derivatives* of the kinematical properties of a particle.

The difficulty can be stated as follows: Fundamentally, the proper-time of a particle is the cumulation of the time-coördinate in its *co-accelerated* coördinate system. To measure the proper-time rate of change of some property of the particle, we should therefore examine the evolution of the property *in this CACS*, and thence take its time-derivative.

But the problem is that the CACS is *not* a Lorentz frame: it is an accelerated system of coördinates. There are therefore two ways that we can proceed. We can either expand our view somewhat, and embrace the complete coördinate-independence employed in the formulation of General Relativity (allowing us to analyse accelerated coördinate systems, although still in a flat spacetime); or we can choose to remain within the realm of Special Relativity, and perform some fancy footwork to extract the physically meaningful quantities for the accelerated particle.

We shall choose the latter path—both because this is the historically favoured option, for electrodynamical considerations in flat spacetime; but moreover because our actual needs, for the purposes of this thesis, are such that we shall only need to delve into the philosophical framework of General Relativity on *two* occasions—this section being the first, and Section 3.3.4 being the second; it is arguably not worth the effort to carry around the full power of General Relativity when our applications are so relatively simple.

Let us therefore consider immediately the problem of computing the *proper-time derivative of the unit four-spin of an accelerated particle*; this example will illustrate the general physical principles completely. Since we are remaining within the analytical framework of Special Relativity, the best that we can do, in an attempt to place ourselves as close as possible to the CACS, is to set up an MCLF for the particle at some instant in time. For simplicity, we label this particular time $t = 0$, for both the CACS and the MCLF, and place the particle, instantaneously at rest, at the spatial origin $\boldsymbol{x} = \boldsymbol{0}$ in both of these coördinate systems. Now, from the definition of the unit four-spin, equation (2.55), we have

$$\Sigma(0) = (0, \boldsymbol{\sigma}); \tag{2.59}$$

and of course the four-velocity of the particle, at this instant in time, is just

$$U(0) = (1, \boldsymbol{0}). \tag{2.60}$$

Now, we know that the CACS will begin to *accelerate away* from the MCLF, as time passes from this instant; we shall therefore need some way to notationally reflect the difference between the components of the four-spin and four-velocity in the two frames. For the purposes of this thesis, we denote the CACS components by placing *parentheses* around the symbol; and the MCLF components by placing *square brackets* around the symbol. Since, at $t = 0$, the CACS and MCLF have been constructed so that the four-spin and four-velocity have identical components, we may therefore write (2.59) and (2.60) in this new notation:

$$\begin{aligned} (U(0)) = [U(0)] &= (1, \mathbf{0}), \\ (\Sigma(0)) = [\Sigma(0)] &= (0, \boldsymbol{\sigma}). \end{aligned} \tag{2.61}$$

Let us consider the particle a small time dt after the instant $t = 0$. Now, as far as the *particle* is concerned, its four-velocity (in its CACS) remains $(1, \mathbf{0})$ for all time, since it is of course always at rest with respect to itself; hence,

$$(U(dt)) = (1, \mathbf{0}),$$

and thus, somewhat trivially,

$$(\dot{U}) = 0. \tag{2.62}$$

On the other hand, the *unit four-spin* of the particle may *change* in its CACS: not in its *zero*-component—since, in the CACS, Σ is always a purely spacelike quantity; rather, the *direction* $\boldsymbol{\sigma}$ of the *three*-spin may of course be *precessing*. Let us refer to this CACS precession of $\boldsymbol{\sigma}$ by simply the symbol $\dot{\boldsymbol{\sigma}}$; the author assures the reader in advance that, for our current purposes, this notation will be found to be quite unambiguous. We thus have, in the CACS,

$$(\Sigma(dt)) = (0,\ \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}\, dt),$$

and thus, as a consequence,

$$(\dot{\Sigma}) = (0, \dot{\boldsymbol{\sigma}}), \tag{2.63}$$

where we are taking as understood the fact that the equation is referring to the instant $t = 0$.

Now let us turn our attention to the description of the particle, at time $t = dt$, in the *MCLF*. The three-velocity of the particle, as seen in the MCLF, is at this time given by

$$\boldsymbol{v}(dt) = \dot{\boldsymbol{v}}\, dt + \mathrm{O}(dt^2), \tag{2.64}$$

where $\dot{\boldsymbol{v}}$ is its three-acceleration in the MCLF at $t = 0$. We may thus apply the Lorentz transformation (G.1) to the four-velocity U, by the three-velocity $\dot{\boldsymbol{v}}\, dt$:

$$[U(dt)] = (1, \dot{\boldsymbol{v}}\, dt) + \mathrm{O}(dt^2),$$

and hence

$$[\dot{U}] = (0, \dot{\boldsymbol{v}}). \tag{2.65}$$

Now, let us pretend, for the moment, that we wished to have a way to connect the rate of change $(\dot{U})$ in the CACS, equation (2.62), with the rate of change $[\dot{U}]$ in the MCLF, equation (2.65). This is of course a hypothetical scenario, since the rate of change of U in the CACS is trivially zero, and hence $(\dot{U})$ is not a quantity that can be in any way useful to us; but nevertheless the procedure is intructive: We could write

$$(\dot{U}) = [\dot{U}] - (0, \dot{\boldsymbol{v}}). \tag{2.66}$$

The connection (2.66) reminds us that, while the CACS and MCLF coïncide at $t = 0$, they *fail* to coïncide, in general, for $t > 0$, and hence time derivatives of the components (U) in the CACS are not equivalent to the time derivatives of the components $[U]$ in the MCLF.

We now apply the same considerations to the four-spin $\varSigma$. In the MCLF, at $t = dt$, we may again apply the Lorentz transformation (G.1), by the velocity $\dot{\boldsymbol{v}}\, dt$ of (2.64), to the components of the four-spin. Before doing so, however, we must again recall that, in the time interval dt, the three-spin $\boldsymbol{\sigma}$

has *precessed* by an amount $\dot{\boldsymbol{\sigma}}\,dt$. (That this amount is the same whether measured in the CACS or MCLF may be recognised, in advance, from the fact that the Thomas precession relation (2.58) given in the previous section does not modify the rate $\dot{\boldsymbol{\sigma}}$ if the initial velocity $\boldsymbol{v}$ linking the two frames is zero; corrections are, in this case, in fact *second* order in dt.) Therefore, we seek to boost the precessed spin,

$$(0,\ \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}\,dt),$$

by the velocity $\dot{\boldsymbol{v}}\,dt$. We find

$$[\Sigma(dt)] = ((\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})dt,\ \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}\,dt) + \mathrm{O}(dt^2),$$

and hence

$$[\dot{\Sigma}] = ((\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}), \dot{\boldsymbol{\sigma}}). \tag{2.67}$$

Now let us connect this MCLF result, (2.67), to the corresponding CACS result, (2.63): Clearly, the former is identical to the latter, *except* for the extra zero-component contribution $(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})$. To subtract off this quantity, we note that

$$[\dot{U}]\cdot\Sigma \equiv [\dot{U}\cdot\Sigma] = -(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}),$$

at the instant $t = 0$; and $U = (1, \mathbf{0})$; hence, we *subtract* $(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})$ from the zero-component by *adding* the term $U[\dot{U}\cdot\Sigma]$:

$$(\dot{\Sigma}) = [\dot{\Sigma}] + U[\dot{U}\cdot\Sigma]. \tag{2.68}$$

Now, in constrast to the case of $\dot{U}$, the quantity $\dot{\Sigma}$ is actually *very important* to our considerations, since it encapsulates the *precession of the three-spin of the particle*. Thus, we must be careful to choose the correct quantity—$(\dot{\Sigma})$ *or* $[\dot{\Sigma}]$—for any mathematical analysis we wish to perform on such a particle. Usually, we shall find that it is the quantity $(\dot{\Sigma})$ that we should in fact be considering, because it represents the precession of the particle *in the CACS*—and hence is independent of any particular coördinate frame we choose to use to evaluate components of quantities. On the other hand, the naïve proper-time derivative of the four-vector Σ^α actually furnishes us with the quantities $[\dot{\Sigma}^\alpha]$:

$$d_\tau[\Sigma^\alpha] \equiv [\dot{\Sigma}^\alpha],$$

because we generally evaluate the components Σ^α in a *Lorentz frame*, and *then take the derivatives of these components*. Thus, the quantity

$$\dot{\Sigma}^\alpha + U^\alpha \dot{U}_\beta \Sigma^\beta$$

that one frequently sees in the literature should, in the author's notation, be surrounded by square brakets—and is indeed simply the right-hand side of (2.68).

We now turn to the question of finding appropriate *names* for the quantities $(\dot{\Sigma})$ and $[\dot{\Sigma}]$, so that they shall not be confused in the remainder of this thesis. If one looks, for example, in Jackson's textbook [113, Sec. 11.11], one finds that he has, indeed, recognised the fundamental importance of $(\dot{\Sigma})$: he has given it a new symbol F^α in his equation (11.166) (to which we shall return shortly). However, he simply refers to it as "the terms with coefficient $(ge/2mc)$ in (11.162)". Now, it may be acceptable to refer to this quantity as "the terms with coefficient $(ge/2mc)$ in (11.162)" on one, or perhaps two, occasions; but if one were to read about "the terms with coefficient $(ge/2mc)$ in (11.162)" repeatedly, one would no doubt appreciate the need for a simpler name. Furthermore, we shall shortly have need to consider proper-time derivatives, similar in nature to "the terms with coefficient $(ge/2mc)$ in (11.162)", but for completely different quantities altogether.

We shall therefore refer to $(\dot{\Sigma})$ as the *covariant proper-time derivative* of Σ; and the quantity $[\dot{\Sigma}]$ as the *partial proper-time derivative* of Σ. The justification for this nomenclature is that it is of course that which is commonly used in General Relativity (*see*, *e.g.*, [153]). Although our description above is in terms of the quantities of Special Relativity, the concepts involved are of course borrowed directly from the General theory.

At this point, we shall make a simplifying definition: for the remainder of this thesis, we recognise the fact that the quantity $(\dot{U}) \equiv 0$ is both trivial and useless, and indeed shall not appear again after this paragraph; but the quantity $[\dot{U}]$ *is* useful, and indeed shall be used quite frequently; hence, we shall allow $[\dot{U}]$ to be denoted by simply the symbol $\dot{U}$, on the understanding that no confusion will result in practice.

We also note that the partial and covariant derivatives are important not just for the four-spin Σ, but in fact for practically *all* proper-time derivatives of manifestly-covariant quantities. For example, we note that $(\dot{\Sigma})$ *itself* is a purely spacelike four-vector in the rest frame of the particle; hence, when we compute *its* proper-time derivative, we must likewise distinguish between the partial and covariant derivatives. And, since this process, for the covariant derivative, always yields *another* purely spacelike four-vector in the rest-frame of the particle, we find that *all covariant derivatives* of Σ must be considered in this way; the final term of (2.68) is essentially the Christoffel symbol term that is "spat out" for each covariant derivative in General Relativity.

As another example of the wide-ranging need for the covariant derivative, we note that, since $\dot{U}$ is *also* a four-vector that is purely spacelike in the rest frame, then *its* derivative, $\ddot{U}$, also has partial and covariant flavours. In particular, we note that

$$(\ddot{U}) \equiv [\ddot{U}] + U\dot{U}^2, \tag{2.69}$$

which *appears* confusing because $\dot{U}$ itself is both acting as the quantity being differentiated, as well as appearing in the "Christoffel symbol" of (2.68). Now, the Abraham term for the *radiation reaction force on a point charge* is often written, manifestly-covariantly, as

$$\Gamma^\mu = \frac{2}{3}\frac{q^2}{4\pi}\left\{\ddot{U}^\mu + U^\mu(\dot{U}_\alpha\dot{U}^\alpha)\right\}.$$

Recalling that the naïve differential $\ddot{U}^\mu$ appearing here is in fact the *partial* derivative $[\ddot{U}]$, we thus see that the Abraham force is in fact simply

$$\varGamma = \frac{2}{3}\frac{q^2}{4\pi}(\ddot{U}). \tag{2.70}$$

The reason for this occurrence of simplicity is that the covariant derivative is of course just the derivative *as seen by the accelerated particle itself*, and is hence the appropriate generalisation of the nonrelativistic result

$$\boldsymbol{F} = \frac{2}{3}\frac{q^2}{4\pi}\ddot{\boldsymbol{v}}. \tag{2.71}$$

We likewise find, for the Thomas–Bargmann–Michel–Telegdi equation [214, 25], that the relativistic generalisation of the nonrelativistic torque $\boldsymbol{N}$ is simply

$$(\dot{S}) \equiv [\dot{S}] + U[\dot{U}\cdot S]. \tag{2.72}$$

Indeed, it is just this quantity that Jackson was referring to in his equation (11.166). If one uses the expressions listed in Section G.4.8 for the components of $(\dot{\varSigma})$ in an *arbitrary* Lorentz frame, one can in fact prove quite quickly Jackson's (11.166):

$$\dot{\boldsymbol{\sigma}} = \frac{1}{\gamma}(\dot{\boldsymbol{\varSigma}}) - \frac{1}{\gamma+1}(\dot{\varSigma})\boldsymbol{v} + \frac{\gamma^2}{\gamma+1}\boldsymbol{\sigma}\times(\boldsymbol{v}\times\dot{\boldsymbol{v}}). \tag{2.73}$$

This equation is most important for one to obtain *Thomas's equation* for the $\dot{\boldsymbol{\sigma}}$ of a magnetic dipole [113, Sec. 11.11]—indeed, to find the precession due to *any* covariant torque $N \equiv (\dot{S})$. It includes—of course!—the Thomas precession term of the previous section, which automatically emerges through the use of relativistic kinematics; this effect is *independent* of the particular torque N that one may wish to consider; but it is of course (indirectly) dependent on the *force* equation of motion, through $\dot{\boldsymbol{v}}$.

2.6.10 The FitzGerald three-spin

In Section 2.6.7, we introduced the unit three-spin $\boldsymbol{\sigma}$, that will be used to represent any internal degrees of freedom for our classical particle that may be fully described in terms a three-direction in its rest frame. We now introduce *another* three-vector quantity, closely related to the three-spin $\boldsymbol{\sigma}$, but with subtle differences that will be found to simplify considerably a number of explicit algebraic expressions that we will encounter in this thesis: the *FitzGerald three-spin*,

$$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{\gamma}{\gamma+1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}. \tag{2.74}$$

The reason for the author naming it after FitzGerald may be appreciated if we compute its three-magnitude:

$$\boldsymbol{\sigma}'^2 = 1 - (\boldsymbol{v}\cdot\boldsymbol{\sigma})^2.$$

The magnitude of $\boldsymbol{\sigma}'$ is (like that of $\boldsymbol{\sigma}$) unity, if $\boldsymbol{\sigma}$ lies in a plane perpendicular to the three-velocity $\boldsymbol{v}$; but it is *contracted* by a factor of

$$\sqrt{1-\boldsymbol{v}^2} \equiv \frac{1}{\gamma}$$

if $\boldsymbol{\sigma}$ lies parallel or antiparallel to the direction of $\boldsymbol{v}$; in other words, $\boldsymbol{\sigma}'$ acts like a *FitzGerald–Lorentz contracted* [87] version of $\boldsymbol{\sigma}$.

The utility (indeed, the *indispensability*) of $\boldsymbol{\sigma}'$ as an algebraic tool will be manifestly clear when we consider the retarded field expressions of Chapter 5.

(The author in fact employed the quantity $\boldsymbol{\sigma}'$ in the published paper of Appendix F [65] as one of a number of "convenient quantities", but did not realise at the time that its simplifying properties are of quite a general nature.)

2.6.11 Relativistic Lagrangian mechanics

There arise the dual questions of whether Lagrangian mechanics can be formulated in a *relativistically meaningful* way, and whether such a formulation can be written in *manifestly covariant* terms.

The former question will be answered in more detail in Section 2.6.12; basically, as long as one can obtain a valid Lagrangian description that is dynamically correct to *first order in the velocity* of the particle, then there is a definite procedure that one may follow to "bootstrap" the resulting equations of motion, unambiguously, into the relativistic domain; the kinematical consequences of Einsteinian mechanics are then automatically included correctly.

If one wishes to actually write down a Lagrangian that itself automatically yields the correct equations of motion to *all orders* in the particle's velocity, but by still using the lab-time t of the Euler–Lagrange equations (2.22), then one is generally advised to begin with the "bootstrap" procedure above, and thence seek a Lagrangian function that reproduces the equations of motion. It is indeed possible to find such functions for simple situations; *e.g.*, for a free particle, one may use [96, Sec. 7–8]

$$L = -m\sqrt{1 - \boldsymbol{v}^2};$$

for an electric charge, one notes that the Lorentz force law is actually correct relativistically, and hence the electromagnetic part of the Lagrangian (2.42) still suffices. However, it is not clear [96] whether such a procedure can be made to work for arbitrary systems of relativistic particles.

The second question above is whether one can find a relativistic Lagrangian framework that is *manifestly covariant*. The subtlety arising here (*see*, *e.g.*, [96, Sec. 7–9]) is that one should seek to use the *proper time* τ as the "time" parameter in the variational framework. For *constant mass* particles, one then faces the problem that the four translational degrees of freedom z^α are actually *constrained* by the identity (2.53):

$$(d_\tau z)^2 = 1; \tag{2.75}$$

hence, there are effectively only three translational degrees of freedom for such systems. On the other hand, for systems of *arbitrary mass*, the four components of the mechanical momentum p^α are clearly *in*dependent, since the identity

$$p^2(\tau) \equiv m^2(\tau)$$

is now a *definition* of the function $m^2(\tau)$, rather than a constraint. But any use of the proper-time τ as the "time parameter" in the variational framework will always lead to the constraint (2.75) for the generalised velocities, and hence cannot yield a formalism embodying four truly independent degrees of freedom.

The way out of this dilemma is to define a *new* time parameter, θ, related differentially to the proper-time by

$$d\theta(\tau) \equiv \frac{d\tau}{m(\tau)}, \tag{2.76}$$

and use this new time parameter as *the* "time" parameter in the Euler–Lagrange formalism. The *generalised velocities* of the translational motion of the particle, in this new formalism, can be obtained by means of the chain rule:

$$\begin{aligned} d_\theta z &\equiv (d_\theta \tau)(d_\tau z) \\ &\equiv m(\tau) U(\tau) \\ &\equiv p(\tau); \end{aligned}$$

hence, in terms of the theta-time, the generalised velocities are simply the components of the *mechanical momentum*:

$$d_\theta z(\theta) \equiv p(\theta).$$

An additional benefit of using the theta-time θ is that *massless* particles may (in principle) be described by the formalism, since both θ and p are well-defined for such particles, but τ and U are not.

The theta-time formalism outlined above appears to be the fundamentally most rigorous way of defining a manifestly covariant Lagrangian or Hamiltonian framework for classical particles. The author is unsure who first developed it; the theta-time is of course a well-known parametrisation of the trajectory of a light beam in General Relativity (*see*, *e.g.*, [184]); and its use as per the above as a rigourisation of the manifestly covariant Lagrangian or Hamiltonian formalism is most definitely known to experts in the field of classical electrodynamics [122], albeit not frequently described in the literature. The author will leave it to other workers to trace the historical precedents of this formalism.

In *practice*, however, one may, for massive particles, embark on a sloppier but somewhat logistically simpler path, that contains the essential physics, keeping always in the back of one's mind that the theta-time formalism may always be dusted off and brought forward to rigourise the results if so required: one may simply employ the proper time τ and four-velocity U in the Lagrangian or Hamiltonian framework, *pretending that the four components of U are not actually constrained by (2.53)*, until the equations of motion themselves are obtained; one then allows the relation (2.53) to again hold true. In using this procedure, we are employing four generalised velocities of translational motion that we are essentially treating as independent, which are in fact *not* independent, but which are acting as proxies for the four generalised velocity coördinates that *are* independent. Thus, for example, we may write down a free-particle Lagrangian of

$$L = \frac{1}{2} m U^2 \tag{2.77}$$

or

$$L = m\sqrt{U^2}, \tag{2.78}$$

pretending in each case that $U^2 \neq 1$, until after we have applied the Euler–Lagrange equations (2.22). Taking the time-parameter to be τ for these equations, one then finds, for either Lagrangian (2.77) or (2.78), the equation of motion

$$d_\tau(mU) = 0,$$

and thus, by the definition of the mechanical momentum p,

$$d_\tau p = 0.$$

We can likewise obtain the equation of motion for an *electric charge*, by simply adding a term to the free-particle Lagrangian:

$$L = \frac{1}{2}mU^2 + q(U\cdot A).$$

The *canonical momentum* four-vector for the electric charge is then directly obtained:

$$b_\alpha \equiv \partial_{U^\alpha} L = mU_\alpha + qA_\alpha;$$

in other words, we again find the *minimal coupling* result:

$$b = p + qA. \tag{2.79}$$

The Euler–Lagrange equation for b then yields

$$d_\tau b - q\partial(U\cdot A) = 0,$$

which, on using (2.79), yields

$$d_\tau p = -qd_\tau A + q\partial(U\cdot A).$$

We now need to use the *relativistic convective derivative*,

$$d_\tau A \equiv (U\cdot\partial)A,$$

to find

$$\begin{aligned} d_\tau p &= -q(U\cdot\partial)A + q\partial(U\cdot A) \\ &\equiv q(\partial\wedge A)\cdot U. \end{aligned}$$

Noting now the definition

$$F \equiv \partial \wedge A,$$

we thus find

$$\dot{p} = qF \cdot U, \tag{2.80}$$

the covariant expression of the Lorentz force law.

Now, to apply the above manifestly covariant Lagrangian formalism to particles possessing "internal" degrees of freedom—which we shall describe using the normalised spin four-vector Σ,—one finds that one has to deal with various new subtleties that are introduced. Firstly, for the Euler–Lagrange equations of motion arising from the *translational* degrees of freedom z^α, we shall find that the normalised four-spin Σ will appear among the factors that need to be subjected to the proper-time derivative operator d_τ. The question then arises as to whether we should use the *partial* or *covariant* proper-time derivative of this quantity. The author confesses that his viewpoint on this subject has changed quite recently: he [65] originally believed that it should be the *covariant* derivative; however, one needs to note that the Lorentz *dot-product*,

$$d_\tau(A \cdot B),$$

must be treated consistently; if one differentiates the *components* of (say) A, then one must of course also differentiate the components of B. This implies a *partial* derivative. (This requirement will be used extensively in Chapter 5.) This point is subtle, for the case of the dipole force law, because the "other quantity" is in fact the *electromagnetic field* $F_{\mu\nu}$, for which we must in fact use the *convective* derivative: a *third* version of the proper-time derivative. This issue was, in fact, only discovered by the author a few days before the printing of this thesis; it is discussed in more depth in Sections 4.2.1 and 4.3.4.

The second issue arising in connection with the use of the four-spin is the question of obtaining the *rotational* equations of motion for the system, *i.e.*, the torque. Clearly, the corresponding Euler–Lagrange equations will be those due to the Euler angle generalised coördinates of the rest-frame three-spin $\boldsymbol{s} \equiv s\boldsymbol{\sigma}$, as in the nonrelativistic case [96]. But relativistic generalisations of these Euler angles do not appear to be worth the complications involved, since they do not even form a *three*-vector in the rest frame [96], let alone a four-vector in the arbitrarily moving frame; thus, any *manifestly covariant* derivation must arguably employ the procedure of constraints [96] on the relativistic generalisation Ω^α of the nonrelativistic angular frequency vector $\boldsymbol{\omega}$, which contains *combinations* of the time-derivatives of the Euler angles. The author has also (despite attempts apparently successful, but then found to be wanting) been unable to formulate any relativistic version of the Euler–Lagrange equations for the *explicit* Euler-angle expressions, in any fashion that might reasonably be described as "transparent".

Fortunately, the torque equation of motion found in Chapter 4 will be simply the *Thomas–Bargmann–Michel–Telegdi equation* [214, 25], which can be derived on quite general physical grounds [108], and indeed has been derived in so many different ways in the literature that it took the author several days just to read them all. Since no modification to this famous and well-loved law will be necessary, we shall be content, for the purposes of this thesis, to simply follow the "relativistic bootstrap" procedure (described in the next section) to obtain the relativistic torque equation of motion from the nonrelativistic result; it is trusted that the reader will be satisfied with this result, even if it might be considered preferable if a manifestly covariant Lagrangian derivation, along the lines described in this section, could be found. We leave the latter as an exercise for the reader!

2.6.12 The "relativistic bootstrap" process

One frequently wishes to obtain equations of motion for a physical system that are relativistically correct, yet without needing to consider all of the (often counter-intuitive) complications involved when the system is moving with an arbitrarily relativistic speed. One therefore often decides to analyse the system in question in the *nonrelativistic limit*; and then, after the correct physics has been ascertained, kinematically "extrapolate" the results to the relativistic domain.

Such a procedure sounds simple enough. It is, however, a potential minefield. In this section, we will describe the steps that need to be taken to ensure that one is not led down an incorrect path; on the other hand, it is shown that, with suitable care, the process is not as difficult as sometimes imagined.

Firstly, it must be noted that Galilean kinematics and Lorentzian kinematics do not fit together as well as one might at first think. On the surface of it, it may appear that the former is the "first approximation" of the latter: Galilean expressions appear to be correct up to first order in a particle's velocity, $\boldsymbol{v}$, but not to second order. For example, the mechanical momentum of a point particle is given, in Galilean kinematics, by

$$\boldsymbol{p} = m\boldsymbol{v},$$

whereas in relativistic kinematics it is given by

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1-\boldsymbol{v}^2}} = m\boldsymbol{v} + \frac{1}{2}m\boldsymbol{v}^2\boldsymbol{v} + \cdots .$$

But the first-order-in-$\boldsymbol{v}$ rule is often *incorrect*. Consider a dumbbell model of an electric dipole: two charges on the ends of a stick. Call their relative displacement $\Delta\boldsymbol{l}$. Relativistically, this relative displacement is actually the three-vector part of a four-vector, ΔL^α, which has vanishing zero-component in the object's rest frame. If this object is boosted by a velocity $\boldsymbol{v}$, then the Lorentz transformation (G.1) shows that, to first order in $\boldsymbol{v}$, the components of the four-vector ΔL are transformed into

$$\begin{aligned} \Delta L^0 &= (\boldsymbol{v}\cdot\Delta\boldsymbol{l}) + \mathrm{O}(\boldsymbol{v}^2), \\ \Delta\boldsymbol{L} &= \Delta\boldsymbol{l} + \mathrm{O}(\boldsymbol{v}^2). \end{aligned} \tag{2.81}$$

The component ΔL^0 is our warning flag: it tells us that, to first order in $\boldsymbol{v}$, the two charges are not only separated in *space*, they are also separated in *time*—even though they are "simultaneous" in their rest frame. This failure of simultaneity is an elementary result in any text on Special Relativity, but due attention is not always brought to the fact this effect is *first order* in the velocity $\boldsymbol{v}$.

How does Galilean kinematics warn of this complication? *It doesn't.* Newton's "universal time" admits of no such manipulation.

It is at about this point that one's faith in nonrelativistic physics generally takes a sharp turn for the worse. *Galilean kinematics is not even correct to first order in $\boldsymbol{v}$, in general.* Those for whom group theory is second nature recognise this simply by virtue of the fact that the transformations of Lorentz group, and those of the Galileo group, are of a completely different inherent structure (*see*, *e.g.*, [237, 24, 111, 97, 98, 133]).

How, then, is one to generalise a valid analysis of nonrelativistic physics into the relativistic domain? The answer is, in one sense, obvious, but on deeper reflection, somewhat subtle: Galilean kinematics *is* correct to *zeroth* order in particle velocities. One need therefore only obtain the physically correct equations of motion for a physical system to zeroth order, convert them to relativistic form, and, *voila!*, one has the answers one seeks. That this is indeed a valid procedure may be recognised by recalling that, if one's equations are valid in some chosen Lorentz frame, and they are written in manifestly covariant form, then they are automatically valid in *all* Lorentz frames.

There is, however, a catch: The equations of Galilean physics are formulated in terms of *three-vectors* and *scalars*. To convert these equations to relativistic form, we need to place these three-vectors and scalars into *appropriate Lorentz quantities*: four-scalars, four-vectors, four-tensors, six-vectors, *etc*. (*See* Section A.8 for a description of the nomenclature used here.) The catch is that, *a priori*, there is no way of fundamentally knowing just *which*

Lorentz quantities should be used: this must be postulated, by someone, at some stage. For example, it is all well and good to know that, to zeroth order, the force on an electric charge q is the Coulomb force:

$$\boldsymbol{F}|_{v=0} = q\boldsymbol{E}|_{v=0}; \tag{2.82}$$

but this does not tell us whether the three-vectors $\boldsymbol{F}$ and $\boldsymbol{E}$ are parts of four-vectors, six-vectors, or quite possibly some higher-rank tensors altogether. Of course, we *know*, now, that $\boldsymbol{F}$ is the three-part of a four-vector, and that $\boldsymbol{E}$ is the electric part of a six-vector; Lorentz, Minkowski and Einstein figured this out for us. But it must always be kept in mind that this knowledge relies on Laws written by our forbears into the Statute Books of Theoretical Physics: it is not in any way derivable from the Galilean expressions.

Once one has appreciated this subtlety, the procedure for "bootstrapping" a nonrelativistic result to the relativistic domain is straightforward. One must simply compute the relevant physics in the frame in which the centre of energy of the system is stationary (if "constituents" in the system are moving, it is assumed that their fully relativistic behaviour is *a priori* known), and then slot these results into the corresponding relativistic structures.

There is, nevertheless, one pitfall into which unfortunate travellers still drop: one must ensure that *all* components of the relativistic structures being used are correctly computed. This seems trivial, but is ignored at one's peril. For example, the electric charge force (2.82) alone does *not* provide sufficient information for one to obtain the relativistic force equation of motion for an electric charge. The reason is a simple matter of arithmetic: equation (2.82) is a three-equation, *i.e.*, it represents three separate component equations, yet we (following our relativistic forbears) know that the relativistic force is a *four-vector*, $\dot{p}^\alpha$. We therefore need a *fourth* equation to supplement (2.82) before we can obtain all four components of $\dot{p}^\alpha$ in this frame. Of course, in the case of an electric charge, this is trivial: the *power into* the charge vanishes, for $\boldsymbol{v} = \mathbf{0}$:

$$P|_{v=0} = 0; \tag{2.83}$$

the postulates of relativity tell us that the power input is, in fact, the zero component $\dot{p}^0$ of $\dot{p}^\alpha$. Equations (2.82) and (2.83), together with the postulate that $\boldsymbol{E}$ is the electric part of the six-vector $F_{\alpha\beta}$, and the definition of the four-velocity (G.3), finally *do* uniquely give us the relativistic expression for the Lorentz force law:

$$\dot{p} = qF \cdot U.$$

The author may seem to be belabouring the point of ensuring all components are computed fully; in the case of the Lorentz force, the (often overlooked) fourth equation required is trivial; but, in general (and the dipoles will be examples of this), this consideration is of the utmost importance.

Finally, we must point out one further complication to this "bootstrap" process, when one wishes to use *Lagrangian* mechanics to obtain the nonrelativistic result. The problem is that, in the course of using the Euler–Lagrange equations, one takes the *derivative* of the Lagrangian with respect to the (generalised) velocities. This operation effectively *decreases the order of* $\boldsymbol{v}$ *in all terms by one power*. Since one usually wishes to perform the "bootstrap" operation on the equations of motion themselves—not the Lagrangian,—one must therefore *retain* an order of $\boldsymbol{v}$ in the initial nonrelativistic Lagrangian treatment, in order that the resulting equations of motion are correct to zeroth order.

On the surface of it, such a procedure would seem to run contrary to our above insistence that Galilean kinematics is *not*, in general, correct to even first order in the velocities. The author submits that this dilemma may arguably be evaded on the grounds of a philosophical technicality: The generalised velocities appearing in Lagrangian treatments are in fact required for the purposes of *dynamical* considerations; the failure of the Galilean framework noted above is in fact a *kinematical* deficiency. It is therefore suggested by the author that, in general, one *may* analyse the Lagrangian of a physical system to first order in the velocities, *as long as side-effects of relativistic kinematics, such as the Thomas precession, are not inserted by hand*; one may then obtain the equations of motion to zeroth order, and insert them into the relativistic structures, without running foul of relativistic invariance; and the side-effects of relativistic kinematics, put to one side in the above procedure, will in fact *emerge automatically* when these final relativistic equations of motion are expanded out, in some given Lorentz frame. To avoid nomenclatorial confusion, the author shall refer to the former, dynamically valid framework described above as the *pre-relativistic limit* of the relativistic equations; it will, in fact, correspond to how the particle itself would describe its evolution, in terms of its co-accelerated coördinate system. (*See* Section A.8.17.) The latter, kinematically expanded-out framework will simply be referred to as the *nonrelativistic limit* of the relativistic equations: this corresponds to how one would describe the motion of the particle, in some given Lorentz frame, when the actual acceleration, *etc.*, of its motion is substituted, where necessary, into the dynamical equations of motion.

(In fact, it will also be necessary to apply the framework of the relativistically rigid body formalism of Chapter 3 to obtain the full equations of motion; this is discussed in Chapter 4.)

The above assertions of the author may appear to dwell unnecessarily on philosophical problems. But the author notes that many capable and reputable physicists have tried to obtain appropriate relativistic equations of motion for particles carrying dipole moments; and many have stumbled and fallen by confusing the *kinematical* effect of the Thomas precession for a *dynamical* effect—and have ended up counting it twice; or have counted it dynamically, but spoilt the integrity of their kinematics in the process; or have forgotten to count it at all. It will be shown in Chapter 4 that, despite the apparent tortuosity of the author's reasoning above, the equations resulting from such an approach do in fact possess all the qualities one would wish of them—not least of which being simplicity.

2.7 Classical limit of quantum mechanics

A thorny question is the following: In what sense does classical mechanics represent the "limit" of quantum mechanics?

The author will not add any new answers to this question. Most physicists will be familiar with Ehrenfest's theorem (*see*, *e.g.*, [150, Ch. VI] and [69, Sec. 31]): the equation of motion for the *expectation value* of an operator is the expectation value of the corresponding classical Hamiltonian equation of motion.

The rub is that, despite first appearances, the expectation values do *not* follow the classical equations of motion, unless the functional dependence of the Hamiltonian equation of motion is such that the expectation value of the function is equivalent to the function of the relevant expectation values. This is only exactly true of a few physical systems (where the Hamiltonian H is a polynomial of the second degree in the q's and b's).

However, regardless of the system being considered, one can generally construct *wave-packets* for which the classical equations of motion are a good approximation—namely, those physical situations for which one may speak with some validity of complementary variables taking on approximately well-defined values (limited by the Heisenberg uncertainty principle) [69]. In such cases, the difference between the taking the expectation of a function of the conjugate variables, and taking the function of the expectations of the conjugate variables, is small compared to the motion of the wave-packet as a whole.

It is simple enough to review Ehrenfest's theorem in theory, as we have briefly done above. It is far more difficult to decide, for any given physical application, just how well the classical equations equations of motion *will* describe the gross motion of the particles under consideration. Clearly, the

Lorentz force equation of motion for charged particles works well in a wide variety of applications; no one would deny its applicability to the real world. Further than this, one must tread carefully.

In the early days of quantum mechanics, it was commonly stated that the newly-invented *spin* degrees of freedom had *no* classical limit whatsoever. But accelerator physicists know this to be an exaggeration: the Thomas–Bargmann–Michel–Telegdi equation of motion [214, 25]—which is a completely *classical* equation—describes the precession of spins to a high degree of accuracy. (Indeed, in the interpretation of the extremely precise measurements of the magnetic moments of fundamental particles, the Thomas–Bargmann–Michel–Telegdi equation is generally used as an analytical tool [19].)

Of course, a classical object *may* indeed possess "spin" angular momentum: it is simply the angular momentum of the object in its rest frame, about its centre of energy; it therefore comes as no surprise that a classical limit should exist. What *is* surprising, to some, is that this limit exists, and is entirely valid, even for *spin-half* particles—the lowest non-zero quantum of spin possible; in other words, it is *not* necessary to invoke the "large quantum number" arguments of the Old Quantum Theory to make contact with the classical formalism. In fact, one finds that, for a single particle, the classical spin precession equation describes accurately the evolution of the three-vector characterising the *expectation value* of the spin: its "latitude" describes the ratio of the amplitudes in the "up" and "down" states; and its "longitude" describes the relative phase between these two amplitudes. The *overall* phase of the wavefunction is—as with any reduction to the classical limit—lost. (*See*, *e.g.*, [155, App. 1.3] for a thorough and rigorous derivation of this correspondence.)

But the successes of the Lorentz force law, and the Thomas–Bargmann–Michel–Telegdi spin precession equation, do not mean that classical equations of motion may be used *carte blanche*. One must always recall that there will be situations in which the wave-packets assumptions fail.

The author will, in the remaining chapters of this thesis, obtain numerous consequences arising from the careful consideration of classical electrodynamics; some of these results are new. The reader may validly ask: In which physical situations may these results be used? The author is not ashamed to admit that he does not precisely know; it is, arguably, a question to which a hasty answer would almost definitely be wrong. The author therefore suggests a "Suck It and See" approach, as has been used, to great advantage, with the Lorentz force and the Thomas–Bargmann–Michel–Telegdi equations. If the equations derived herein by the author describe your physical application well, then count your blessings: you have a new analytical tool available to you.

2.8 Pointlike trajectory parametrisation

Consider a pointlike particle. We shall, in later chapters, have need to parametrise its (relativistic) path, $z(\tau)$, around some particular event, which we shall refer to as the *zero event*. The following question then arises: What is the best way to perform this parametrisation?

Consider, first, Lorentz invariance: no matter which Lorentz frame we use to perform our computations, the final, Lorentz-invariant results must be the same. Choice of an arbitrary Lorentz frame gives us ten degrees of freedom that we may choose at will: the origin of the four spacetime coördinates; the three-velocity of the frame relative to that of the particle, at the zero event; and the three Euler angles describing the spatial orientation of the frame.

Clearly, the kinematical motion of the particle provides "natural" choices for seven of these degrees of freedom, that will clearly simplify the mathematics considerably: namely, setting the origin of spacetime to be at the zero event (we also set the origin of τ to be at this same event); and setting the relative three-velocity of the Lorentz frame and the particle's motion to be zero at this same event. We shall leave the spatial orientation of the Lorentz frame arbitrary.

2.8.1 Covariant Taylor series expansion

We now make the following assumption: *the worldline of the particle*, $z(\tau)$, *is an analytical function of* τ. Without this assumption, it is practically impossible to proceed; we shall, however, leave philosophical questions on this topic aside, for the purposes of this thesis.

Expanding $z^\alpha(\tau)$ as a Taylor series about $\tau = 0$, we have

$$z^\alpha(\tau) \equiv c_0^\alpha + c_1^\alpha \tau + c_2^\alpha \tau^2 + c_3^\alpha \tau^3 + c_4^\alpha \tau^4 + c_5^\alpha \tau^5 + c_6^\alpha \tau^6 + \mathrm{O}(\tau^7), \qquad (2.84)$$

where the c_i^α are constants, dependent on the physical motion around $\tau = 0$. (*See* Section A.3.17 for a description of the $+\mathrm{O}(\,)$ notation.)

The keeping of terms up to sixth order in τ in (2.84) is not an arbitrary choice: the considerations of this thesis require precisely this many orders be retained, and no more.

2.8.2 Redundancies in the covariant parametrisation

By setting the origin of spacetime to be at the zero event, we have set

$$c_0^\alpha = 0$$

in (2.84); the choice of zero velocity at $\tau = 0$ likewise sets

$$c_1^\alpha = (1, \mathbf{0}).$$

However, it is clear that the manifestly covariant parametrisation (2.84) still contains a greater number of parameters than is required to fully specify the path of the particle. To see this, one need only recall that, once the three-position $\boldsymbol{z}(\tau)$ of the particle is specified for all proper time, the *lab-time* of the particle, $t(\tau)$, is *automatically* specified, by virtue of the identity

$$dt \equiv \frac{d\tau}{\sqrt{1 - \boldsymbol{v}^2(\tau)}}$$

(*see* equation (A.58) of Section A.8.19). In other words, only *three* of the four components of $z^\alpha(\tau)$ are independent, and hence the parametrisation (2.84) contains a redundant parameter in each order of τ.

To remove these redundant parameters, it would clearly be sufficient to eliminate the zero-components c_i^0, in favour of the spatial components $\boldsymbol{c}_i$.

2.8.3 Non-covariant parametrisation

While mathematically quite acceptable, the $\boldsymbol{c}_i$ of equation (2.84) do not, however, have a direct connection with one's intuitive understanding of the motion of a pointlike particle. Clearly, any *other* set of three-vectors, that are in one-to-one correspondence with the $\boldsymbol{c}_i$, will serve the same purpose, mathematically.

The author suggests that the most natural parametrisation of the path, that removes all redundant parameters, is in terms of the *lab-frame motion* of the particle. In other words, we consider the three-space position of the particle, as seen in a particular lab frame, as a function of the time coördinate in this frame:

$$\boldsymbol{z}(t) = \boldsymbol{z}|_0 + \boldsymbol{v}|_0 t + \frac{1}{2}\dot{\boldsymbol{v}}|_0 t^2 + \frac{1}{6}\ddot{\boldsymbol{v}}|_0 t^3 + \frac{1}{24}\dddot{\boldsymbol{v}}|_0 t^4 + \frac{1}{120}\ddddot{\boldsymbol{v}}|_0 t^5 + \mathrm{O}(t^6), \tag{2.85}$$

where the overdots on the non-covariant three-vector $\boldsymbol{v}$ denote d_t (*see* Sections A.3.10, A.3.18, A.8.15 and A.8.20), and our choice of Lorentz frame sets

$$\boldsymbol{z}|_0 = \boldsymbol{0}$$

and

$$\boldsymbol{v}|_0 = \boldsymbol{0}.$$

The Taylor series (2.85) possesses the dual advantages that it corresponds to what we would have written down as *the* trajectory of the particle before the advent of Special Relativity—and hence is in somewhat more contact with our intuition than a manifestly-covariant expression,—while still containing exactly the right number of free parameters to specify the fully relativistic path of the particle.

2.8.4 Conversion of parametrisation

To make connection between the parametrisations (2.84) and (2.85), we first differentiate the former with respect to τ,

$$d_\tau z^\alpha(\tau) \equiv c_1^\alpha + 2c_2^\alpha \tau + 3c_3^\alpha \tau^2 + 4c_4^\alpha \tau^3 + 5c_5^\alpha \tau^4 + 6c_6^\alpha \tau^5 + \mathrm{O}(\tau^6), \tag{2.86}$$

and the latter with respect to t:

$$\boldsymbol{v}(t) = \dot{\boldsymbol{v}}t + \frac{1}{2}\ddot{\boldsymbol{v}}t^2 + \frac{1}{6}\dddot{\boldsymbol{v}}t^3 + \frac{1}{24}\ddddot{\boldsymbol{v}}t^4 + \mathrm{O}(t^5), \tag{2.87}$$

where in (2.87), and hereafter, we take the notation $|_0$ to be *understood* for the quantities $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, $\dddot{\boldsymbol{v}}$ and $\ddddot{\boldsymbol{v}}$, if they are not adorned to the contrary with an explanatory subscript. Now, from the definition of τ, namely,

$$d\tau^2 \equiv dz^\alpha dz_\alpha,$$

we immediately find the constraint

$$(d_\tau z^\alpha)(d_\tau z_\alpha) \equiv \dot{z}^\alpha \dot{z}_\alpha \equiv 1. \tag{2.88}$$

Using (2.86) directly yields

$$\begin{aligned}\dot{z}^\alpha \dot{z}_\alpha &= c_1^2 + 4(c_1 \cdot c_2)\tau + \Big\{6(c_1 \cdot c_3) + 4c_2^2\Big\}\tau^2 + \Big\{8(c_1 \cdot c_4) + 12(c_2 \cdot c_3)\Big\}\tau^3 \\ &\quad + \Big\{10(c_1 \cdot c_5) + 16(c_2 \cdot c_4) + 9c_3^2\Big\}\tau^4 \\ &\quad + \Big\{12(c_1 \cdot c_6) + 20(c_2 \cdot c_5) + 24(c_3 \cdot c_4)\Big\}\tau^5 + \mathrm{O}(\tau^6);\end{aligned}$$

thus, to satisfy (2.88), we require

$$c_1^2 = 1,$$
$$4(c_1 \cdot c_2) = 0,$$

$$
\begin{aligned}
6(c_1\cdot c_3) + 4c_2^2 &= 0,\\
8(c_1\cdot c_4) + 12(c_2\cdot c_3) &= 0,\\
10(c_1\cdot c_5) + 16(c_2\cdot c_4) + 9c_3^2 &= 0,\\
12(c_1\cdot c_6) + 20(c_2\cdot c_5) + 24(c_3\cdot c_4) &= 0.
\end{aligned} \tag{2.89}
$$

We now divide the temporal component $d_\tau z^0$ of (2.86) into the spatial components $d_\tau \boldsymbol{z}$, to obtain

$$
\begin{aligned}
\boldsymbol{v}(\tau) &\equiv d_t \boldsymbol{z}(\tau)\\
&\equiv \frac{d_\tau \boldsymbol{z}(\tau)}{d_\tau t(\tau)}\\
&\equiv \frac{\boldsymbol{c}_1 + 2\boldsymbol{c}_2\tau + 3\boldsymbol{c}_3\tau^2 + 4\boldsymbol{c}_4\tau^3 + 5\boldsymbol{c}_5\tau^4 + 6\boldsymbol{c}_6\tau^5 + \mathrm{O}(\tau^6)}{c_1^0 + 2c_2^0\tau + 3c_3^0\tau^2 + 4c_4^0\tau^3 + 5c_5^0\tau^4 + 6c_6^0\tau^5 + \mathrm{O}(\tau^6)}.
\end{aligned} \tag{2.90}
$$

To proceed from here, one needs to perform the division (2.90) term-by-term; at each step, one needs to make use of the identities (2.89), and then replace τ wherever it appears in favour of t (by reverting the expression already found, to the preceding order, for t as a function of τ). The result at each step is compared to the parametrisation (2.87), to replace the c_i^α by the quantities $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, $\dddot{\boldsymbol{v}}$ and $\ddddot{\boldsymbol{v}}$. The procedure is straightforward, but tedious; the author will spare the reader the gory details. One finally finds that the correct parametrisation is

$$
\begin{aligned}
t(\tau) &= \tau + \frac{1}{6}\dot{\boldsymbol{v}}^2\tau^3 + \frac{1}{8}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\tau^4 + \frac{1}{120}\Big\{13\dot{\boldsymbol{v}}^4 + 3\ddot{\boldsymbol{v}}^2 + 4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\Big\}\tau^5\\
&\quad + \frac{1}{144}\Big\{(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) + 2(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + 27\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\Big\}\tau^6 + \mathrm{O}(\tau^7),
\end{aligned} \tag{2.91}
$$

$$
\begin{aligned}
\boldsymbol{z}(\tau) &= \frac{1}{2}\dot{\boldsymbol{v}}\tau^2 + \frac{1}{6}\ddot{\boldsymbol{v}}\tau^3 + \frac{1}{24}\Big\{\dddot{\boldsymbol{v}} + 4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\Big\}\tau^4\\
&\quad + \frac{1}{120}\Big\{\ddddot{\boldsymbol{v}} + 10\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + 15(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}\tau^5 + \mathrm{O}(\tau^6).
\end{aligned} \tag{2.92}
$$

(It will be noted that (2.91) contains terms up to sixth order in τ, whereas (2.92) only contains terms up to fifth order. This order of expansion has been chosen by the author to yield only those orders of terms required for this thesis, and no more; the reason that (2.91) requires an extra order is that its *leading*-order term is of order τ, whereas that of (2.92) is of order τ^2, and hence when we form powers of these expressions via the binomial theorem, the cross-terms between the leading and last-retained orders will both be of order τ^7. This also explains why we retained terms up to sixth order in τ in (2.84)—as it encompasses all four components (2.91) and (2.92),—but only terms up to fifth order in (2.85), since the latter only affects the spatial components (2.92).)

2.8.5 Verification of parametrisation

As a method of verification, one may simply *start* with the results (2.91) and (2.92)—derivation unseen,—and verify that they do indeed satisfy the requirements laid out previously. (The computer algebra program RADREACT, of Appendix G, does precisely this.) Taking the τ-derivative of (2.91) and (2.92), one obtains

$$\begin{aligned}\gamma(\tau) \equiv d_\tau t(\tau) \;&=\; 1 + \frac{1}{2}\dot{\boldsymbol{v}}^2\tau^2 + \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\tau^3 + \frac{1}{24}\Big\{13\dot{\boldsymbol{v}}^4 + 3\ddot{\boldsymbol{v}}^2 + 4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\Big\}\tau^4 \\ &\quad + \frac{1}{24}\Big\{(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) + 2(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + 27\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\Big\}\tau^5 + \mathrm{O}(\tau^6), \qquad (2.93)\\ d_\tau\boldsymbol{z}(\tau) \;&=\; \dot{\boldsymbol{v}}\tau + \frac{1}{2}\ddot{\boldsymbol{v}}\tau^2 + \frac{1}{6}\Big\{\dddot{\boldsymbol{v}} + 4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\Big\}\tau^3 \\ &\quad + \frac{1}{24}\Big\{\ddddot{\boldsymbol{v}} + 10\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + 15(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}\tau^4 + \mathrm{O}(\tau^5). \qquad (2.94)\end{aligned}$$

Our first check comes from the identity (2.88): for the expressions (2.93) and (2.94), we find

$$\dot{z}^\alpha\dot{z}_\alpha \equiv (d_\tau t)^2 - (d_\tau\boldsymbol{z})^2 = 1 + \mathrm{O}(\tau^6).$$

Now computing $\boldsymbol{v}(\tau)$ from (2.90), we find

$$\boldsymbol{v}(\tau) \;=\; \dot{\boldsymbol{v}}\tau + \frac{1}{2}\ddot{\boldsymbol{v}}\tau^2 + \frac{1}{6}\Big\{\dddot{\boldsymbol{v}} + \dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\Big\}\tau^3$$

$$+\frac{1}{24}\Big\{\dddot{\boldsymbol{v}}+4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}+3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}\tau^4+\mathrm{O}(\tau^5), \tag{2.95}$$

and reverting $t(\tau)$ from (2.91), one finds

$$\begin{aligned}\tau(t) &= t-\frac{1}{6}\dot{\boldsymbol{v}}^2t^3-\frac{1}{8}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})t^4-\frac{1}{120}\Big\{3\dot{\boldsymbol{v}}^4+3\ddot{\boldsymbol{v}}^2+4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\Big\}t^5\\ &-\frac{1}{144}\Big\{(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}})+2(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})+6\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\Big\}t^6+\mathrm{O}(t^7);\end{aligned} \tag{2.96}$$

substituting (2.96) into (2.95), we are returned to the definition (2.87); this is the second check. Finally, we note that $\gamma(\tau)$, computed in (2.93) as $d_\tau t(\tau)$, may alternatively be computed via

$$\gamma(\tau)\equiv\frac{1}{\sqrt{1-\boldsymbol{v}^2(\tau)}}; \tag{2.97}$$

using (2.95), we are returned to the expression (2.93).

Thus, even without seeing the explicit derivation of (2.91) and (2.92), one knows that they are, in fact, a correct parametrisation of the path of the point particle, around its instantaneous-rest event.

2.8.6 Spin degrees of freedom

We now consider the case in which the point particle possesses three internal degrees of freedom constituting a *spin vector*, $\boldsymbol{\sigma}$, in its rest frame. As described in Section 2.6.7, this three-vector generalises to the four-vector Σ when the particle is in arbitrary relativistic motion. Clearly, we could consider a manifestly-covariant parametrisation of $\Sigma(\tau)$ *à la* that of $z(\tau)$, namely,

$$\Sigma(\tau)=c'_0+c'_1\tau+c'_2\tau^2+c'_3\tau^3+c'_4\tau^4+\mathrm{O}(\tau^5), \tag{2.98}$$

but from the discussion of Section 2.8.2, we already know that this will introduce a *redundant* parameter in each order, since the four-spin Σ must satisfy the constraint

$$(\Sigma\cdot U)=0.$$

Thus, we again seek a set of *three*-vector parameters that will parametrise the evolution of the spin $\Sigma(\tau)$ without redundancy.

Clearly, the set of three-vectors $\boldsymbol{c}'_n$ of (2.98) would serve this purpose, but, again, they are not, in the author's opinion, the most natural or intuitive choices. Instead, we shall, following the discussion of Section 2.8.3, parametrise the *three-spin* $\boldsymbol{\sigma}$, in terms of its *lab-time* evolution. The analogue of (2.87) is clearly

$$\boldsymbol{\sigma}(t) \equiv \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}} t + \frac{1}{2}\ddot{\boldsymbol{\sigma}} t^2 + \frac{1}{6}\dddot{\boldsymbol{\sigma}} t^3 + \frac{1}{24}\ddddot{\boldsymbol{\sigma}} t^4 + \mathrm{O}(t^5), \tag{2.99}$$

where we again understand the unadorned quantities $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\sigma}}$, $\ddot{\boldsymbol{\sigma}}$, $\dddot{\boldsymbol{\sigma}}$ and $\ddddot{\boldsymbol{\sigma}}$ to denote the spin and its derivatives *evaluated at* $t = 0$.

Unlike the analysis of Section 2.8.4, the conversion of the lab-time parametrisation (2.99) into a *proper*-time parametrisation is simple, since we already have $t(\tau)$, equation (2.91): inserting this into (2.99), we find

$$\begin{aligned}\boldsymbol{\sigma}(\tau) \;\equiv\; & \boldsymbol{\sigma} + \dot{\boldsymbol{\sigma}}\tau + \frac{1}{2}\ddot{\boldsymbol{\sigma}}\tau^2 + \frac{1}{6}\Big\{\dddot{\boldsymbol{\sigma}} + \dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}\Big\}\tau^3 \\ & + \frac{1}{24}\Big\{\ddddot{\boldsymbol{\sigma}} + 4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}} + 3(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\Big\}\tau^4 + \mathrm{O}(\varepsilon^5).\end{aligned} \tag{2.100}$$

The similarities between this equation for $\boldsymbol{\sigma}(\tau)$, and (2.95) for $\boldsymbol{v}(\tau)$, are obvious; they are both obtained by substituting $t(\tau)$ into the lab-time Taylor expansions (2.87) and (2.99).

Chapter 3

Relativistically Rigid Bodies

We envisage a rigid sphere—that is, a body possessing spherical form when examined at rest—of radius R, . . .

—— A. Einstein (1905) [75]

3.1 Introduction

The Special Theory of Relativity was only four sections old when those words were written (translated here [78] from the German, of course). Einstein's seminal work [75] is arguably one of the most carefully thought-out papers in the history of Physics. If he himself used a relativistically rigid sphere, then it must be an acceptable construct.

If only it were that simple. Numerous myths have arisen, over the years, about what the theory of relativity does and does not say about the notion of rigidity. It will be necessary for the author to bring the fictional nature of some of these old wives' tales to the attention of the reader, as the author would like to use just such an object in his considerations.

We shall also derive quantitative relations for the constituents of such rigid bodies, that will be of vital importance in the following chapters.

3.2 Notions of rigidity

In this section, we briefly review the philosophical assumptions underlying Galilean (Section 3.2.1) and Einsteinian (Section 3.2.2) notions of rigidity.

3.2.1 Galilean rigidity

The *Galilean* concept of rigidity—namely, that the three-positions of the constituents of the rigid body maintain a fixed three-geometrical relationship with respect to one another, up to three-rotations—clearly is nonsensical in a Lorentz world: it is not formulated in Lorentz-covariant terms.

There are two ways in which one can view the problem. The first is to invoke the *FitzGerald–Lorentz contraction* [87]: as the body's velocity increases, the body should effectively "contract" in length along the direction of its motion. The Galilean model would require extra, contrived forces to bring about such a contraction.

The second way to view the problem—which the author overwhelmingly prefers—is to start with a stationary body. According to Galilean mechanics, a boost by a velocity $\boldsymbol{v}$ of this system does not affect the three-geometrical relationships between the constituents: they all move coherently, with the same velocity, and the same separation vectors between them, as when the body is static. But according to *relativistic* mechanics, we find that a boost by velocity $\boldsymbol{v}$ should actually take us into a new *Lorentz* frame, which *mixes* the temporal and spatial components of the original frame. Thus, if the three-geometry is fixed in one Lorentz frame, then it will not, in general, be so in another.

The reason that the second way of looking at the problem is preferred by the author is that the former, "contraction" argument clings somewhat to the pre-relativistic concept of Newton's "universal time": it "measures" the lengths of objects *over a constant-time hypersurface of the measurer*, rather than considering the rest frame of the object being *measured* of fundamental importance. Of course, the difference is that the rest frame of a moving particle is *"tilted"* in spacetime relative to the frame in which it is seen to move with velocity $\boldsymbol{v}$; the "contraction" notion obfuscates the presence of this "tilt". (Indeed, if the "ends" of the body are measured *at constant proper time for the body*, their three-separation is actually *lengthened*, not contracted, as the Lorentz transformation shows; it is the presence of the corresponding difference in *time* between the two "ends" of the body that ensures that its invariant length is a constant.)

Of course, the concept of the "FitzGerald–Lorentz contraction" played an important rôle in the development of the relativistic theory, and is given due historical respect by the author; but nevertheless one should begin weeding it out of one's fundamental conceptual framework, since it is, in itself, not a fully-ripened concept, and can lead one on quite erroneous paths of logic if applied recklessly.

3.2.2 Einsteinian rigidity

We now turn to Einstein's concept of rigidity, as illustrated by the quote at the head of this chapter: namely, that the body is always of some predetermined three-geometrical shape *in its instantaneous rest-frame*. The author cannot improve on Pearle's concise and accurate review [168] of this topic, one of the many instructive sections in his 1982 review of classical electron models, and so will reproduce it here verbatim (for logistical simplicity, citation references to the author's Bibliography are added in square brackets):

> Fermi (1922) [83, 84, 85], Wilson (1936) [242], Kwal (1949) [128], and Rohrlich (1960) [178] found another way to construct a relativistically invariant theory that is more in keeping with Abraham's [1, 2] original economical notion of avoiding discussion of mechanical forces. The electron is to move in such a way that a

> Lorentz frame can always be found in which the electron is instantaneously at rest and spherically shaped. The equation of motion
>
> $$m_0 a^\mu = f^\mu_{\text{self}} + f^\mu_{\text{ext}}$$
>
> is required to hold always in this instantaneous rest frame. In this formulation, the mechanical force is assumed capable of maintaining the electron's spherical shape in the electron's rest frame, while making no net contribution to the total self-force. Thus f^μ_{self} is totally electromagnetic, and the electromagnetic energy and momentum are computed in such a way as to satisfy the correct relativistic relationship.
>
> We shall call this the *relativistically rigid electron*. The nonrelativistic notion of rigidity, which requires unchanging dimensions of an object regardless of its state of motion or the observer's reference frame, is impossible because of the Lorentz contraction. In order to extend the notion of rigidity to relativity, it is best to talk of a rigid *motion* of an object, rather than a rigid *object* (Pauli, 1958). One can define the rigid motion of an object *with respect to a point* as a motion in which the dimensions of the object are fixed in the instantaneous rest frame of the point. In this sense, the electron undergoes rigid motion with respect to its center. We shall only discuss nonrotative motion of the electron, where the surface of the electron is at rest in the rest frame of its center.

Change the word "surface" to "volume" in this last sentence (Pearle considers only spherical *shell* charge densities; we shall choose to consider *uniform* spherical densities), and this extract sums up the author's approach to this problem quite accurately.

We also note Pearle's proof of his equation (9.15),

$$\frac{d}{d\tau}P^{\nu}_{\text{mech}} = \frac{d}{d\tau}m_0 v^{\nu}(\tau),$$

(*see* pp. 265–6 of [168]), that states that, for a relativistically rigid body, the contribution, to the equation of motion, of the mechanical stress-energy maintaining the "rigid motion" of the body is just that of the inertial term of the corresponding mechanical energy in the rest frame.

For further philosophical arguments in favour of relativistically rigid bodies as appropriate bases for our physical considerations, we refer the reader to the literature cited above. The diagrams in Pearle's review [168] are also of great benefit in visualising the issues involved, and will be referred to in the following sections.

3.3 Trajectories of rigid body constituents

We now turn to the question of obtaining explicit, relativistically correct expressions for the *trajectories of the constituents* of a relativistically rigid body.

We begin, in Section 3.3.1, by explaining just what we mean by the term "constituent", and we establish appropriate notation to label such objects. In Section 3.3.2, we ease the reader into the algebraic considerations to follow, by obtaining the constituent trajectories assuming *Galilean* rigidity. We then, in Section 3.3.3, repair the relativistic deficiencies of the Galilean results, following the guidelines of Section 3.2.2. In Section 3.3.4, we point out a simple yet vitally important consequence of the relativistic constituent trajectorial parametrisation, that is a familiar face from another area of classical physics, but which has not often been seen around the traditional haunts of classical electrodynamics. In Section 3.3.5, we compute the trajectories of the constituents in terms of the lab-time t, rather than the body's proper time τ. Finally, in Section 3.3.6, we define the spins of the constituents of the body in terms of the spin of the body as a whole.

3.3.1 What is a "constituent"?

If we consider a body which has some finite three-geometry in its rest frame, we need to have a way of describing the parts of the body making up the whole. We shall call these the *constituents* of the body.

If a body is constructed from discrete constituents, then we can name them individually, and thereafter refer to them by name; but if the body is (as it will be for the considerations of this thesis) formed from a *continuum* of constituents, then we need some other way of identifying them individually.

Consider the body in its instantaneous rest frame. Let us place the *mechanical centre of energy* of the body at the origin of coördinates. We can then identify any constituent by its *relative three-position*, $\boldsymbol{r}$, in this rest frame.

It may seem that we should really be talking about not a single point at the position $\boldsymbol{r}$ as a "constituent", but rather a *small elementary volume* surrounding the position $\boldsymbol{r}$, since, rigorously speaking, a mere point in a uniformly distributed region does not actually contain anything. But we do not particularly care that the "point" contains "nothing", since we know that we shall, ultimately, be integrating over an *infinite number* of "points",—which *does* yield "something"; these are simple concepts of elementary calculus. Thus, we shall continue to refer to the point $\boldsymbol{r}$ as "a constituent"; the reader may, if they like, place a solid stone at this point if they feel the conceptual need to do so.

We shall also consider the three-vector $\boldsymbol{r}$ to serve as a suitable *name* for the constituent at the point $\boldsymbol{r}$. For example, we shall refer to "the constituent $\boldsymbol{r}$", as if "$\boldsymbol{r}$" were a name like Tom or Dick. This will avoid an unnecessary amount of circumlocution in the remainder of this thesis. Mathematical properties of the constituent $\boldsymbol{r}$ are likewise labelled by the symbol $\boldsymbol{r}$; or, more frequently, by the subscript $_r$ on the quantity in question. (Note that pure LaTeX cannot boldface an italicised subscript, and its boldfacing of roman subscripts are of incorrect size; we therefore use the bare Roman subscript $_r$, on the understanding that it is to be taken to be the three-vector $\boldsymbol{r}$.)

Of course, the use of the rest-frame three-position $\boldsymbol{r}$ *at all* would be somewhat ambiguous if the distribution of the constituents were not *spherically symmetric* around the centre of energy, because an instantaneous rest frame is only defined up to an arbitrary rotation. However, we shall only consider *spherical bodies of uniform constituent density* in this thesis, and hence shall not need to worry about such subtleties.

3.3.2 Constituent trajectories for Galilean rigidity

We now consider the problem of obtaining the trajectories of the constituents of the rigid body. From the discussion of Section 3.2.2, we know that the trajectory of the *centre of energy* of the body—in our case, the centre of the sphere, which we shall henceforth simply refer to as "the centre"—is to be taken as *the trajectory of the body considered as a whole*. The trajectories of the *other* constituents must be formed in such a way so as to maintain the spherical rest-frame three-geometry of the body.

For simplicity, we again choose our Lorentz frame according to the considerations of Section 2.8, so that the *centre* of the body is instantaneously at rest at the "zero event". Thus, at the instant $\tau = 0$, the four-position of the constituent $\boldsymbol{r}$ is simply given by

$$\begin{aligned} t_r(0) &= 0, \\ \boldsymbol{z}_r(0) &= \boldsymbol{r}. \end{aligned} \tag{3.1}$$

Now let us first see how we would proceed if we were to assume *Galilean* rigidity to hold true, rather than Einsteinian rigidity. Employing Newton's universal time, we would then find

$$t_r(t)\big|_{\text{Galilean}} = t, \tag{3.2}$$

$$\boldsymbol{z}_r(t)\big|_{\text{Galilean}} = \boldsymbol{z}(t) + \boldsymbol{r}, \tag{3.3}$$

where we assume, following the discussion of Section 3.2.2, that the body does *not* rotate at any time. The first result, (3.2), simply states that simultaneity is universal in Galilean physics. The second result, (3.3), states that the absolute position of the constituent $\boldsymbol{r}$ is given by the vectorial sum of its relative position $\boldsymbol{r}$ relative to the centre of the body, and the absolute position of the centre of the body itself, $\boldsymbol{z}(t)$.

The lack of relativistic invariance of the Galilean result (3.3) is highlighted most clearly if we compute the relative four-positional offset between the constituent $\boldsymbol{r}$ and the centre of the body:

$$\mathit{\Delta} z_r^\alpha(\tau) \equiv z_r^\alpha(\tau) - z^\alpha(\tau). \tag{3.4}$$

For the Galilean result (3.3), we have

$$\begin{aligned} \mathit{\Delta} t_r(t) &= 0, \\ \mathit{\Delta} \boldsymbol{z}_r(t) &= \boldsymbol{r}, \end{aligned} \tag{3.5}$$

which is clearly *not* transforming as a four-vector, as the velocity of the particle increases with time, as it should.

3.3.3 Constituent trajectories for Einsteinian rigidity

We found, in the previous section, that the essential failing of the Galilean results (3.2) and (3.3) is that the relative position $\mathit{\Delta} z_r^\alpha(\tau)$ does not transform as a four-vector. It is tempting to correct this defect by merely transforming (3.5) according to the Lorentz transformation (G.1),

$$\begin{aligned} \mathit{\Delta} t_r(\tau) &= \gamma(\tau)\Big(\boldsymbol{u}(\tau)\cdot\boldsymbol{v}(\tau)\Big), \\ \mathit{\Delta} \boldsymbol{z}_r(\tau) &= \boldsymbol{u}(\tau) + \frac{\gamma^2(\tau)}{\gamma(\tau)+1}\Big(\boldsymbol{u}(\tau)\cdot\boldsymbol{v}(\tau)\Big)\boldsymbol{v}(\tau), \end{aligned} \tag{3.6}$$

where we set

$$\boldsymbol{u}(\tau) = \boldsymbol{r} \tag{3.7}$$

because we do not wish the body to rotate. This is, indeed, the transformation that the author had used, until after the computer algebra programs of Appendix G had been completed and the final equations of motion obtained (*see* Section G.2.1). However, (3.7) is, in fact, *incorrect*, for the following subtle reason: The non-covariant three-vector $\boldsymbol{u}$ is, according to (3.7), constant in time:

$$\dot{\boldsymbol{u}} = \boldsymbol{0}. \tag{3.8}$$

But we know, from Section 2.6.8, that the three-vector $\boldsymbol{u}$ should *Thomas precess*, as seen from the (fixed) lab frame, as the velocity and acceleration of the body increase from zero, so that the corresponding $\boldsymbol{u}$ measured in the CACS does *not* precess. In other words, by using the transformation (3.6) with (3.7), we would, in fact, be unwittingly specifying that the body should start to *rotate*—as seen by the particle itself, in its CACS,—in such a way so that, when this rest-frame rotation is added to the Thomas precession in the lab frame, the net result would be zero.

This rest-frame rotation would violate the very assumptions underlying the construction of the relativistically rigid body (*see* Section 3.2.2), and hence the conclusions originally drawn by the author from the use of (3.7) were rendered invalid. Somewhat surprisingly, the *final* equations of motion obtained by the author (to be described in Chapter 6) were *unchanged* by his correction of this oversight (to be described below)—even though all expressions up to the penultimate step *were* affected. The author has no clear understanding of this phenomenon as yet; it is discussed further in Section 6.9.

Returning to the derivation under consideration, it is clear that the correct way to compute the trajectory of the constituent $\boldsymbol{r}$ is to *add* the required

Thomas precession term to the otherwise undesirable result (3.8):

$$\dot{\boldsymbol{u}}(\tau) = \frac{\gamma^2(\tau)}{\gamma(\tau)+1}\,\boldsymbol{u}(\tau)\times\Big(\boldsymbol{v}(\tau)\times\dot{\boldsymbol{v}}(\tau)\Big)\,. \tag{3.9}$$

We can convert the lab-time derivative $\dot{\boldsymbol{u}} \equiv d_t\boldsymbol{u}$ into a proper-time derivative by multiplying by γ:

$$d_\tau\boldsymbol{u}(\tau) = \frac{\gamma^3(\tau)}{\gamma(\tau)+1}\,\boldsymbol{u}(\tau)\times\Big(\boldsymbol{v}(\tau)\times\dot{\boldsymbol{v}}(\tau)\Big)\,. \tag{3.10}$$

Since the right-hand side of (3.10) depends itself on $\boldsymbol{u}(\tau)$, we now consider $\boldsymbol{u}$ as a Taylor series in τ, and integrate (3.10) order-by-order. This calculation is performed, and explicitly cross-checked, by the computer algebra program RADREACT, in Section G.6.3 of Appendix G. The result found there (simplified somewhat, notationally, by the author) is

$$\begin{aligned}\boldsymbol{u}(\tau) \;&= \boldsymbol{r} + \frac{1}{12}\boldsymbol{r}\times(\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}})\tau^3 + \frac{1}{24}\boldsymbol{r}\times(\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{v}})\tau^4 \\ &\quad + \frac{1}{240}\boldsymbol{r}\times\Big\{3\dot{\boldsymbol{v}}\times\ddddot{\boldsymbol{v}} + 2\ddot{\boldsymbol{v}}\times\dddot{\boldsymbol{v}} + 19\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}\Big\}\tau^5 + \mathrm{O}(\tau^6).\end{aligned} \tag{3.11}$$

We can now substitute the corrected result (3.11) for $\boldsymbol{u}(\tau)$ into the Lorentz transformation (3.6), and then evaluate this transformation for the parametrisation of the centre of the body derived in Section 2.8. Using the definition

$$z_r^\alpha(\tau) = z^\alpha(\tau) + \mathit{\Delta} z_r^\alpha(\tau), \tag{3.12}$$

we then find we find that

$$\begin{aligned}t_r(\tau) \;&= \Big[1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\tau + \frac{1}{2}(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\tau^2 + \frac{1}{6}\Big\{\Big[1 + 4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}^2 + (\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\Big\}\tau^3 \\ &\quad + \frac{1}{24}\Big\{\Big[3 + 13(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big](\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + (\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}}) + 12\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\Big\}\tau^4 + \mathrm{O}(\tau^5),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\boldsymbol{z}_r(\tau) \;&= \boldsymbol{r} + \frac{1}{2}\Big[1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}\tau^2 + \frac{1}{6}\Big\{\Big[1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\ddot{\boldsymbol{v}} + 2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}\tau^3 \\ &\quad + \frac{1}{24}\Big\{\Big[1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dddot{\boldsymbol{v}} + \Big[4 + 13(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + 3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + 3(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}\tau^4\end{aligned}$$

$$+\frac{1}{120}\Big\{\Big[1+(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dddot{\boldsymbol{v}}+2\Big[5+14(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}$$
$$+5\Big[3+11(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big](\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+52\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+6(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\Big\}\tau^5+\mathrm{O}(\tau^6). \qquad (3.14)$$

3.3.4 Accelerative redshift

The constituent $\boldsymbol{r}$ has, in the above, been treated as a part of the overall body, and has thus had its motion described in terms of the *body's* proper time, τ. On the other hand, the constituent often needs to be considered as an individual particle in its own right. The proper time τ_r of this particle $\boldsymbol{r}$ is defined solely by its four-trajectory z_r^α; the body proper time τ is, in this context, merely a convenient quantity that parametrises this path via equations (3.14) and (3.13). Now, by definition,

$$d\tau_r^2 \equiv dt_r^2 - d\boldsymbol{z}_r^2;$$

thus,

$$d_\tau\tau_r \equiv \sqrt{(d_\tau t_r)^2-(d_\tau\boldsymbol{z}_r)^2}. \qquad (3.15)$$

Computing (3.15) from (3.14) and (3.13), one finds that

$$d_\tau\tau_r = \Big[1+(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]+(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\tau+\frac{1}{2}\Big\{(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})+3\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big\}\tau^2$$
$$+\frac{1}{6}\Big\{(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})+9\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})+10(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\Big\}\tau^3+\mathrm{O}(\tau^4). \qquad (3.16)$$

This result (3.16) is, in fact, of *major importance* to the considerations of this thesis. Let us explain why. To compute the power into, force on and torque on the relativistically rigid body, we shall need to sum up these quantities for all of the constituents of the body. Let us, for example, consider the *power and force* on the body; the torque follows in the same manner. Now, the four-force on the body as a whole, $\dot{p}^\alpha$, is of course defined in terms of *the proper-time of the body as a whole*:

$$\dot{p} \equiv d_\tau p.$$

On the other hand, the four-force on a *constituent* $\boldsymbol{r}$ is given by some Law of Nature—such as the Lorentz force law—that involves *the proper-time of that constituent*:

$$d_{\tau_r} p_r. \tag{3.17}$$

The contribution of the four-force on this constituent to the mechanical four-momentum of the body as a whole must therefore be computed by means of the *chain rule*:

$$d_\tau p_r \equiv (d_{\tau_r} p_r)(d_\tau \tau_r), \tag{3.18}$$

where it will be noted that the derivative operator on the left-hand side is d_τ, *not* d_{τ_r}. The first factor on the right-hand side of (3.18) is simply the four-force on the constituent, (3.17), as given by the appropriate Law of Nature; whereas the *second* factor of (3.18) is the quantity computed in (3.16). Now, at first sight, it might be thought that, by analysing the body in its instantaneous rest frame, we would find the body and constituent times evolving at the same rate. *But (3.16) shows that this is not true*: at $\tau = 0$, it tells us that

$$d_\tau \tau_r \big|_{\tau=0} = 1 + (\boldsymbol{r} \cdot \dot{\boldsymbol{v}}). \tag{3.19}$$

This result may seem perplexing at first. However, upon further reflection, it can be recognised as just the familiar *gravitational redshift formula*. Of course, from the Equivalence Principle, we know that a gravitational field is simply equivalent to an acceleration, and vice versa; thus, it comes as no surprise that, when a relativistically rigid body *is* accelerated—by *whatever* means—the constituents in this accelerated frame experience the phenomenon of redshift. (Of course, Einstein originally [77] *began* with the accelerated system, and then inferred the gravitational redshift by invoking the Principle of Equivalence; it is curious that the latter is now more rigorously applied than the former.) We shall therefore, for clarity, actually refer to the result (3.19) as the *accelerative redshift formula*, as being a more accurate name for our current purposes.

An intuitive understanding of this result may also be gleaned from Figure 6 of Pearle's review [168]: Due to the acceleration of the particle, the contant-proper-time hypersurfaces are successively more *tilted* as the particle moves away from $\tau = 0$. Thus, constituents in the direction of the acceleration see $d_t\tau < 1$ (since the τ "ticks" are spaced further apart, relative to the t "ticks", for these constituents); constituents in the opposite direction see $d_t\tau > 1$ (since the τ "ticks" are more compressed for these constituents); and constituents perpendicular to the acceleration direction see $d_t\tau = 1$. Noting that $d_t\tau_r \equiv 1$ for $\boldsymbol{v}_r = \boldsymbol{0}$, and taking the reciprocal of (3.19), we find that this intuitive explanation contains the physics completely.

The net result of the above considerations, as far as the remainder of this thesis is concerned, is, therefore, the following: the *body* proper-time derivative of any covariant quantity Q is related to the *constituent* proper-time derivative of this same quantity, at $\tau = 0$, by means of the relation

$$\begin{aligned} d_\tau Q &\equiv \Big[1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big] d_{\tau_r} Q \\ &\equiv \lambda d_{\tau_r} Q, \end{aligned} \tag{3.20}$$

where we have defined the symbol

$$\lambda \equiv 1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}}), \tag{3.21}$$

which will be used extensively throughout the rest of this thesis.

We also note the phenomenon of the *accelerative horizon*: if $(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) < -1$, the constituent $\boldsymbol{r}$ will need to go into *antiparticle motion* to satisfy the rigid body constraints. This can be avoided by keeping the body's size small enough so that all of its constituents are within the horizon produced by the maximum acceleration encountered in the practical situation in question; this will be discussed in more detail in Chapter 6.

3.3.5 Lab-time constituent trajectories

We now obtain the trajectory of the constituent $\boldsymbol{r}$ as $\boldsymbol{z}_r(t_r)$, *without* reference to the body τ at all, by eliminating τ between equations (3.14) and (3.13). Reverting (3.13), we have [4, eq. 3.6.25]

$$\begin{aligned}
\tau(t_r) \;=\; & \frac{1}{\lambda}t_r - \frac{1}{2\lambda^3}(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})t_r^2 \\
& + \frac{1}{6\lambda^5}\Big\{-\lambda(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}}) - \lambda\Big[1+4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}^2 + 3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2\Big\}t_r^3 \\
& + \frac{1}{24\lambda^7}\Big\{-\lambda^2(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}}) - \lambda^2\Big[3+13(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big](\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - 15(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^3 \\
& \qquad - 2\lambda\Big[1-14(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\Big]\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + 10\lambda(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\Big\}t_r^4 \\
& + \mathrm{O}(t_r^5).
\end{aligned} \tag{3.22}$$

Using (3.22) in (3.14), we thus find that

$$\begin{aligned}
\boldsymbol{z}_r(t) \;=\; & \boldsymbol{r} + \frac{1}{2\lambda}\dot{\boldsymbol{v}}t^2 + \frac{1}{6\lambda^3}\Big\{\lambda\ddot{\boldsymbol{v}} - (\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\Big\}t^3 \\
& + \frac{1}{24\lambda^5}\Big\{\lambda^2\dddot{\boldsymbol{v}} - \lambda(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 3\lambda\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 3\lambda(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + 3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\Big\}t^4 \\
& + \frac{1}{120\lambda^7}\Big\{\lambda^3\ddddot{\boldsymbol{v}} - \lambda^2(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 6\lambda^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - 4\lambda^2(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& \qquad - 12\lambda^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - 8\lambda^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 10\lambda^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& \qquad + 15\lambda(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 10\lambda(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& \qquad + 30\lambda\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 15(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}\Big\}t^5 + \mathrm{O}(t^6),
\end{aligned} \tag{3.23}$$

where it is understood that t means t_r. Equation (3.23) gives the trajectory of the constituent $\boldsymbol{r}$ as an individual particle, without reference to the system proper-time τ.

3.3.6 Spin degrees of freedom

We finally consider the spin degrees of freedom of the particle. The various constituents $\boldsymbol{r}$ of the body each have a unit spin vector of their own, $\boldsymbol{\sigma}_r$, which varies with time. We deem that *the spins of the constituents are all aligned in the instantaneous rest frame of the particle*. Thus

$$\boldsymbol{\sigma}_r(\tau) \equiv \boldsymbol{\sigma}(\tau), \tag{3.24}$$

where, again, it is the proper time of the body as a whole that appears here.

We shall defer an explicit expansion of the result (3.24) in terms of the constituent time t_r to Chapter 6; if one desires it for an arbitrary rigid body, one simply needs to substitute (3.22) for $\tau(t_r)$ in (3.24).

Chapter 4

Dipole Equations of Motion

The next stage would naturally be to seek the second approximation terms in equations (4.11) of motion of the centre of the electron. There does not, however, appear to be any definite way of finding them as they depend on the assumptions made as to the constitution of the electron.

—— L. H. Thomas (1927) [214]

4.1 Introduction

He may be accused of being a doubting Thomas; but his doubts were completely well-founded. The exact classical force equation of motion that one obtains for an electron depends on one's assumptions as to the classical nature of its *magnetic dipole moment*. Change your assumptions—while still keeping the same magnetic moment—and your expressions change before your very eyes.

Now, in the past thirty-five years, various workers have found that the standard *textbook* expression for the force on a magnetic dipole possesses a number of properties that are somewhat less than desirable, if one would like to apply them to elementary particles. These workers have also generally realised what Thomas already did in 1927: that *different* models of magnetic dipoles yield *different force laws*. Indeed, one of these alternatives has most *desirable* properties—but unfortunately it is derived from a model that is manifestly and unsalvageably unphysical.

On the other hand, a number of illustrious physicists who contemplated this problem showed, by simple yet powerful *gedanken* experiments, that the simple textbook force expression, while obtained by quite rigorous methods, could not be the whole story: some truth was being hidden somewhere. They therefore analysed the systems in question in meticulous detail; and—as elementary as it may be to us now as Conan Doyle always had it to dear Watson—the puzzle was solved.

Unfortunately, due to the mist of time, it has sometimes become somewhat unclear what exactly the puzzle was in the first place; and whose investigation it was that solved it; or indeed whether perhaps different solutions should apply in different situations. And then the facts of the case were muddled a little further, with spurious characters, apparently fitting the *modus operandi* of the culprit, being rounded up for good measure, and made to stand in the ultimate line-up of publication by peer review; but in fact the wrong man was accused, having inadvertantly switched identities with the culprit, in the minds of the Authorities, decades ago.

If the situation sounds confusing, good. It is. The author began his Ph.D. considering this problem; and, perhaps fittingly, has only obtained a satisfactory understanding of all of the nuances of the situation in the closing days of his candidature. The full story is as complicated as the metaphorical situation above, so we shall, in this chapter, let it unfold slowly, and carefully; it was to assist in this task that Chapter 2 was written in the careful style that it was. By the end of this chapter, we shall have led the reader through the maze of contradictory literature on this topic, and shall emerge with the fully relativistic equations of motion for particles carrying dipole moments—of *arbitrary* nature—that would no doubt satisfy the wishes of all workers in this field.

It should be noted that a summary of the equations of motion obtained in this chapter has been published [65]; this paper is included verbatim in Appendix F. However, it was found by the author, shortly before the printing of this thesis, that an extra term needs to be added to the results of the published paper; this is discussed in detail in Section 4.2.1.

4.2 Newtonian mechanics

In this section, we analyse various classical dipoles from the viewpoint of Newtonian mechanics.

4.2.1 The electric dipole

The simplest dipole to consider is an *electric dipole*: a separation of positive and negative electric charge. Its behaviour is easily derivable, and is not subject to any controversy whatsoever.

Let us choose to analyse what is arguably the simplest model of a fixed electric dipole: a positive charge glued to one end of a rigid stick, with a negative charge of the same magnitude glued to the other end of the stick. (We only consider fixed moments in this thesis, as are applicable for the intrinsic moments of spin-half particles; "induced" moments are not, in general, considered.)

The author shall describe how the following analysis can be made *absolutely* rigorous, according to the relativistically rigid body formalism described in Chapter 3, at the end of this section.

We shall, as with all considerations of this thesis, consider only *pointlike particles* in this chapter; but we shall of course start off with an object that is of finite size, and then take the point limit at the end of the calculations.

Now, an electric dipole moment is fundamentally described by a three-vector, $\boldsymbol{d}$, in the rest frame of the particle. The *direction* $\hat{\boldsymbol{d}}$ of $\boldsymbol{d}$ describes the direction in which the positive-charge end of our rigid stick is pointing;

the *magnitude* d of $\boldsymbol{d}$ describes the strength of the electric dipole moment. We shall make use of the unit three-vector $\boldsymbol{\sigma}$ of Section 2.6.7, via

$$\boldsymbol{\sigma} \equiv \hat{\boldsymbol{d}},$$

to simplify notationally the analysis of the electric dipole.

Let us call the positive charge on our stick q_1, and the negative charge q_2, and likewise label the kinematical quantities of these charges by the subscripts $_1$ and $_2$. For simplicity, let us place the origin of our spatial coördinates at the midpoint of the rigid stick, at the instant $t = 0$ at which we wish to analyse the particle. Let us say that the stick is of length ε, which we shall shrink to zero at the end of the analysis. The positive charge q_1 is thus, according to these definitions, located at the position

$$\boldsymbol{z}_1 = \frac{1}{2}\varepsilon\boldsymbol{\sigma}, \tag{4.1}$$

at $t = 0$; and the negative charge q_2 is located at the position

$$\boldsymbol{z}_2 = -\frac{1}{2}\varepsilon\boldsymbol{\sigma}. \tag{4.2}$$

By convention, the definition of the magnitude d of the dipole moment is such that our charges q_1 and q_2 take the values

$$q_1 \equiv -q_2 \equiv \frac{d}{\varepsilon}. \tag{4.3}$$

Let us now consider the *motion* of the charges q_1 and q_2, in the nonrelativistic limit. Each charge will obviously partake in the motion, $\boldsymbol{v}$, of the centre of the dipole (the mid-point of the rigid stick); but the charges will also have *equal and opposite* contributions to their velocity if the stick is *rotating* (or "precessing", as it is generally referred to). This precession is described by the three-vector $\dot{\boldsymbol{\sigma}}$, the rate of change of the direction of the dipole. Clearly, since $\boldsymbol{\sigma}$ is of constant magnitude (*i.e.*, the dipole moment is fixed in magnitude), we have the identity

$$(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\sigma}}) \equiv 0.$$

By using the elementary result $v = r\dot{\theta}$ for circular motion, and noting that the direction of $\dot{\boldsymbol{\sigma}}$ is the direction of this extra velocity for q_1, and antiparallel to it for q_2, we thus find

$$\begin{aligned} \boldsymbol{v}_1 &= \boldsymbol{v} + \frac{1}{2}\varepsilon\dot{\boldsymbol{\sigma}}, \\ \boldsymbol{v}_2 &= \boldsymbol{v} - \frac{1}{2}\varepsilon\dot{\boldsymbol{\sigma}}. \end{aligned} \tag{4.4}$$

We now consider the external electric and magnetic fields that are acting on our electric dipole. Clearly, since the dipole will eventually be shrunk to infinitesimal size, it is appropriate for us to expand the fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{B}(\boldsymbol{r})$ as three-dimensional Taylor series around the position of the centre of the dipole:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) &= \boldsymbol{E}(\boldsymbol{0}) + (\boldsymbol{r}\cdot\nabla)\boldsymbol{E}(\boldsymbol{0}) + \mathrm{O}(\boldsymbol{r}^2), \\ \boldsymbol{B}(\boldsymbol{r}) &= \boldsymbol{B}(\boldsymbol{0}) + (\boldsymbol{r}\cdot\nabla)\boldsymbol{B}(\boldsymbol{0}) + \mathrm{O}(\boldsymbol{r}^2). \end{aligned} \tag{4.5}$$

Let as refer to $\boldsymbol{E}(\boldsymbol{0})$ and $\boldsymbol{B}(\boldsymbol{0})$ as simply $\boldsymbol{E}$ and $\boldsymbol{B}$ in the following, and likewise for the gradients of these fields, on the understanding that these quantities and derivatives are to be evaluated at the position of the centre of the dipole. Using (4.1) and (4.2) in (4.5), we thus find

$$\begin{aligned} \boldsymbol{E}_1 &= \boldsymbol{E} + \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E} + \mathrm{O}(\varepsilon^2), \\ \boldsymbol{E}_2 &= \boldsymbol{E} - \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E} + \mathrm{O}(\varepsilon^2), \\ \boldsymbol{B}_1 &= \boldsymbol{B} + \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B} + \mathrm{O}(\varepsilon^2), \\ \boldsymbol{B}_2 &= \boldsymbol{B} - \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B} + \mathrm{O}(\varepsilon^2). \end{aligned} \tag{4.6}$$

We now have all the ingredients required to find the pre-relativistic equations of motion for our electric dipole. The force on each electric charge q_1 or q_2 is of course simply the Lorentz force (2.17); the *power into* each charge is likewise the Lorentz result (2.18). Let us consider first the Lorentz force. We shall compute the net force on the electric dipole as a whole as simply the *sum of the forces on the two charges* (the rigorously relativistic consideration of this problem shall be presented shortly):

$$\boldsymbol{F} = q_1\Big\{\boldsymbol{E}_1 + \boldsymbol{v}_1\times\boldsymbol{B}_1\Big\} + q_2\Big\{\boldsymbol{E}_2 + \boldsymbol{v}_2\times\boldsymbol{B}_2\Big\}.$$

Using (4.3), (4.4) and (4.6), one finds

$$\boldsymbol{F} = (\boldsymbol{d}\cdot\nabla)(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) + \dot{\boldsymbol{d}}\times\boldsymbol{B}, \tag{4.7}$$

where we have reïnstated the three-vectors $\boldsymbol{d}$ and $\dot{\boldsymbol{d}}$ into this final expression, and where we have taken the point limit, so terms of order ε or higher vanish.

Let us examine the result (4.7) for the force on an electric dipole in some detail. Firstly, we see that there is a *gradient force* on a stationary dipole,

$$(\boldsymbol{d}\cdot\nabla)\boldsymbol{E}; \tag{4.8}$$

this is of course intuitively understandable, since each charge has a Lorentz force of $q\boldsymbol{E}$; and the electric dipole is essentially a replacement of the Dirac delta function source density of a monopole by a spatial *gradient* of a Dirac delta function, in the direction of $\hat{\boldsymbol{d}}$. The second term in (4.7) is clearly the first relativistic correction to the rest-frame $\boldsymbol{E}$:

$$\boldsymbol{E} \longrightarrow \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B};$$

we shall "bootstrap" up a fully relativistic expression, with a greater deal of rigour, shortly.

We now turn to the *final* term of the electric dipole force equation (4.7), namely,

$$\dot{\boldsymbol{d}}\times\boldsymbol{B}. \tag{4.9}$$

This force may at first be surprising. It applies to a *stationary* electric dipole, but is dependent on its *rate of precession*. However, this force is in fact quite easily understood when one looks at the construction of our electric dipole: if the rigid stick is precessing, then the two charges have velocities that are *equal and opposite* (in circular motion around the centre of the stick, effectively), of order ε. But the *values of their charges*, q_1 and q_2, are *also* equal and opposite, and, moreover, are of order ε^{-1}; thus, the quantity $q\boldsymbol{v}$ for each of these charges is of the *same* sign, and is of order ε^0. But this $q\boldsymbol{v}$ simply couples to the magnetic part of the Lorentz force law, $q\boldsymbol{v}\times\boldsymbol{B}$; hence, we are led to the force (4.9).

If we refer to the part (4.8) of the electric dipole force as the "gradient force", then we clearly need to invent a name for the contribution (4.9). The author considers the term *precession force* to be a suitable name, since the force is fundamentally a coupling of the precession of the dipole to the external magnetic field. A more colourful name might be the term *helicopter force*, which evokes images of the particle "taking off" due to its rigid stick flaying around in the "air" of the magnetic field. Another suitable term might be the *electric dipole Anandan force*, because the magnetic dipole dual of (4.9), highlighted and emphasised by Anandan, has proved controversial in the last couple of years. Anandan was not the first to discover this force, of course; but historical precedent suggests that the person or persons who *draw most attention* to a phenomenon should have the effect named after them: for example, Aharonov and Bohm were not the first to suggest the Aharonov–Bohm effect [74]; Aharonov and Casher were not the first to suggest the Aharonov–Casher effect [107, 10]; *etc.* In any case, the author shall, for the purposes of this thesis, stick to the conservative term "precession force" for (4.9).

Let us now consider the *power into* the electric dipole. We shall again simply add the powers into each electric charge, according to the Lorentz power expression (2.18),

$$P = q_1(\boldsymbol{v}_1\cdot\boldsymbol{E}_1) + q_2(\boldsymbol{v}_2\cdot\boldsymbol{E}_2);$$

we shall make the analysis relativistically rigorous shortly. Again using (4.3), (4.4) and (4.6), one finds

$$P = (\dot{\boldsymbol{d}} \cdot \boldsymbol{E}) + (\boldsymbol{d} \cdot \nabla)(\boldsymbol{v} \cdot \boldsymbol{E}). \tag{4.10}$$

Now, the second term of (4.10) is simply the first-order change in kinetic energy of the particle due to the gradient force (4.8). The *first* term of (4.10), however, is a little more worrying. It tells us that *a stationary electric dipole may increase its mechanical energy*; in relativistic terms, this means that its *mass*—its mechanical rest-energy—may change. The fundamental source of this power may be recognised by following the same line of reasoning as was used to understand the precession force above: in this case, the precession $q\boldsymbol{v}$ of each charge can couple to the external *electric* field, via the Lorentz power expression $q\boldsymbol{v} \cdot \boldsymbol{E}$.

To see whether this "precession power" will *actually* cause the mechanical rest-energy of the dipole to increase, we first have to obtain an equation of motion for the precession itself. This leads us to the question of the *torque* on the electric dipole. Now, each electric charge has no "intrinsic" torque on it, *i.e.*, no torque about the position of the charge itself (this is an unstated but eminently reasonable postulate of Newtonian electrodynamics); but we *do* of course have a torque on the dipole as a whole, due to the fact that the *forces* on the charges are not through the centre of the dipole, but are rather offset: we have a "moment-arm" type of torque

$$\boldsymbol{N} = \boldsymbol{z}_1 \times \boldsymbol{F}_1 + \boldsymbol{z}_2 \times \boldsymbol{F}_2.$$

Again using (4.3), (4.4) and (4.6), one finds

$$\boldsymbol{N} = \boldsymbol{d} \times (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}). \tag{4.11}$$

This is of course the well-known result; the second term is again the first-order relativistic correction to the rest-frame electric field $\boldsymbol{E}$, which we shall rigourise shortly.

Now, the torque on an object (around its centre of energy) is simply the time rate of change of its mechanical *spin angular momentum*,

$$\boldsymbol{N} \equiv \dot{\boldsymbol{s}},$$

in the same way that the force is the time rate of change of its linear mechanical momentum,

$$\boldsymbol{F} \equiv \dot{\boldsymbol{p}}.$$

In general, the spin angular momentum $\boldsymbol{s}$ of a classical object may be of any value, and point in any direction. In particular, there is no good reason why it should point in the same direction as the electric dipole moment: it *can* do so (*e.g.*, if one uses the rigid stick above as an axis for a rotating flywheel, and one assumes the charges and the rigid stick to be themselves massless); but it need not, in general.

On the other hand, we shall often wish to apply the equations of motion derived in this thesis to the case of a *spin-half particle*, such as an electron. For such a particle, we *know* that the mass must be a constant of the motion (namely, m_e); and we *know* that the magnitude s of its spin angular momentum must be a constant of the motion (namely, $\frac{1}{2}\hbar$). But we also know that *any dipole moment of a spin-half particle must be parallel to its spin vector* $\boldsymbol{s}$ (basically because there are no other three-vectors available in the Dirac algebra):

$$\begin{aligned} \boldsymbol{d} &\equiv d\boldsymbol{\sigma}, \\ \boldsymbol{s} &\equiv s\boldsymbol{\sigma} \equiv \frac{1}{2}\hbar\boldsymbol{\sigma}. \end{aligned}$$

With such a parallelism identity in effect, we *may* use the torque result $\boldsymbol{N}$ to obtain the precession rate very simply:

$$\dot{\boldsymbol{d}}\Big|_{d\|s} = \frac{d}{s}\dot{\boldsymbol{s}} \equiv \frac{d}{s}\boldsymbol{N}. \tag{4.12}$$

Now, let us consider such a spin-half particle, in its rest frame. The power into the dipole, (4.10), is then

$$P = (\dot{\boldsymbol{d}} \cdot \boldsymbol{E});$$

using the relation (4.12), we may now write this as

$$P = \frac{d}{s}(\boldsymbol{N} \cdot \boldsymbol{E}).$$

But if we substitute the rest-frame torque $\boldsymbol{N}$ from (4.11), we find

$$P = \frac{d}{s}\boldsymbol{E} \cdot \boldsymbol{d} \times \boldsymbol{E} \equiv 0.$$

Thus, for electric dipoles for which $\boldsymbol{d}$ and $\boldsymbol{s}$ *are* parallel, we find that *the mass of the dipole is a constant of the motion.* This is a non-trivial result, and will *not* be found to be true for all systems considered in this chapter.

We also need to consider the rate of change of the *spin magnitude* s. Differentiating the definition

$$s^2 \equiv \boldsymbol{s}^2,$$

we have

$$\dot{s} \equiv \frac{1}{s}(\boldsymbol{s} \cdot \dot{\boldsymbol{s}}) \equiv \frac{1}{s}(\boldsymbol{s} \cdot \boldsymbol{N}).$$

In the general case, the electric dipole torque (4.12) yields some finite $\dot{s}$; but, again, if the dipole moment $\boldsymbol{d}$ and spin $\boldsymbol{s}$ are *parallel*, we find

$$\dot{s}\big|_{d \| s} = 0.$$

We now consider the problem of providing a rigorous relativistic derivation of the power, force and torque expressions found in this section. We use the framework described in Chapter 3. Clearly, we need to place the dipole as a whole at rest, to use the framework described there; the simultaneity of the two ends of the stick, assumed above, is then appropriate and relativistically correct. We must also multiply all expressions by the accelerative redshift factor $\lambda(\boldsymbol{r})$, which for each of our two charges q_1 and q_2 is simply given by

$$\begin{aligned}\lambda_1 &= 1 + \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}}),\\ \lambda_2 &= 1 - \frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}}).\end{aligned} \tag{4.13}$$

For the computation of the power expression, we note that the *velocities* of the charges (in the dipole rest frame) are already of order ε; the factor of ε^{-1} in the values $\pm d/\varepsilon$ of the charges cancel this factor of ε; hence, there are therefore no "spare" factors of ε^{-1} left which could couple to the latter terms of (4.13). Hence, the power expression is given by the rest-frame value of (4.10) found above:

$$P|_{v=0} = (\dot{\boldsymbol{d}}\cdot\boldsymbol{E}). \tag{4.14}$$

The rigorously relativistic computation of the *force* expression, however, is somewhat more subtle. Here, we find that the electric force on each charge *may couple* with the second factor of (4.13):

$$\boldsymbol{F}_{\text{extra}} = \left(\pm\frac{d}{\varepsilon}\right)\left(\pm\frac{1}{2}\varepsilon(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}})\right)\boldsymbol{E},$$

where the upper (lower) signs apply to charge q_1 (q_2). We thus find an extra *redshift force*,

$$\boldsymbol{F}_{\text{redshift}} = (\boldsymbol{d}\cdot\dot{\boldsymbol{v}})\boldsymbol{E}.$$

Thus, the total rest-frame force on the electric dipole is in fact given by

$$\boldsymbol{F}|_{v=0} = (\boldsymbol{d}\cdot\nabla)\boldsymbol{E} + (\boldsymbol{d}\cdot\dot{\boldsymbol{v}})\boldsymbol{E} + \dot{\boldsymbol{d}}\times\boldsymbol{B}.$$

Now, in the interests of honesty, the author must confess that this extra "redshift force" was only discovered by him a few days before the final printing and binding of this thesis. It has ramifications for the radiation reaction calculations of Chapter 6; fortunately, the author *was* able to insert this additional force (and its corresponding "moment-arm" torque) into the program

RADREACT of Appendix G; the expressions found there and in Chapter 6, and the author's comments on them in Chapter 6, have been updated to reflect the resulting changes to the radiation reaction equations of motion. In fact, this additional contribution *simplified* the final equations of motion somewhat; and it also *removed one term that was clearly unphysical*, which had been causing the author serious concern. Due to this increase in physical correctness of the radiation reaction calculations, the author most definitely considers the redshift force to be physically correct; he only wishes he had obtained it earlier. However, to the author's knowledge, this redshift force has not been obtained in the literature to date; but then again neither has the accelerative redshift factor apparently been used *at all*, for the purposes of electrodynamical considerations.

If the radiation reaction calculations yielded a bonanza with the redshift force, the published paper [65] listed in Appendix F did not fare quite so well. The derivation therein is now incomplete; and therefore the final "uncoupled" equations of motion, listed there, are incorrect. The reason for the author failing to obtain the redshift force from the *manifestly covariant* analysis presented there will be discussed in somewhat more detail in Section 4.3.4; basically, the author inappropriately used the *covariant* proper-time derivative, rather than the *partial* proper-time derivative, on one occasion: the difference between the two leads, somewhat remarkably, to the extra redshift force derived above!

We now turn to the relativistically rigorous derivation of the *torque* equation of motion, in the rest frame. In this case, we note that the quantity $\boldsymbol{z}$ that is crossed into the force expression to obtain the "moment-arm" torque is in fact of order ε, so, as with the power equation of motion, there are no terms of order ε^{-1} remaining that could couple to the second terms in the redshift factors (4.13). Hence the rest-frame torque expression is still that given by (4.11)

$$\boldsymbol{N}|_{v=0} = \boldsymbol{d}\times\boldsymbol{E}. \qquad (4.15)$$

We can now place these results into the appropriate Lorentz quantities. The power and force expressions are of course to be placed into the quantity $\dot{p}$, and the torque expression is to be placed into the quantity $(\dot{S})$. Firstly, we note that the equation

$$\dot{p} = (d\cdot\partial)F\cdot U + [F\cdot\dot{d}\,] \tag{4.16}$$

has the correct properties: to see this, we note that, for $\boldsymbol{v} = \boldsymbol{0}$, the following are true:

$$\begin{aligned}(d\cdot\partial) &= (\boldsymbol{d}\cdot\nabla),\\ F\cdot U &= (0, \boldsymbol{E}),\\ [F\cdot\dot{d}\,] &= (\dot{\boldsymbol{d}}\cdot\boldsymbol{E},\ (\boldsymbol{d}\cdot\dot{\boldsymbol{v}})\boldsymbol{E} + \dot{\boldsymbol{d}}\times\boldsymbol{B}),\end{aligned}$$

where we have used the identities listed in Section G.4 of Appendix G frequently. (The published paper [65] incorrectly uses $(F\cdot\dot{d}\,)$ rather than $[F\cdot\dot{d}\,]$.) We furthermore note that

$$(\dot{S}) = -\widetilde{F}\cdot d - U(d\cdot\widetilde{F}\cdot U) \tag{4.17}$$

similarly has the correct properties, since

$$\begin{aligned}\widetilde{F}\cdot d &= (\boldsymbol{d}\cdot\boldsymbol{B},\ -\boldsymbol{d}\times\boldsymbol{E}),\\ (d\cdot\widetilde{F}\cdot U) &= -\boldsymbol{d}\cdot\boldsymbol{B};\end{aligned}$$

the second term of (4.17) therefore serves to remove the (unwanted) zero-component of the first. The result (4.17) is, of course, simply the manifestly covariant *Bargmann–Michel–Telegdi equation* [25] for the electric dipole.

Finally, it should be noted that, although we employed a "two charges on a stick" model of an electric dipole for the purposes of this section, the results obtained are of quite a general nature, for any fixed dipole $\boldsymbol{d}$ arising from the permanent separation of positive and negative electric charge.

4.2.2 The magnetic-charge dipole

Magnetic monopoles are incompatible with a simple Lagrangian description of the electromagnetic field; they cannot be described by a unique four-potential $A(x)$. They have never been observed experimentally. All known magnetic dipole moments have been shown [115] to most definitely *not* arise through the presence of magnetic monopoles. The author does not believe that magnetic monopoles exist in our Universe.

But Newtonian mechanics does not care for any of this: it does not use the four-potential; it does not worry about what has and has not been discovered by mankind; and it most definitely couldn't care less about the author's opinions. Moreover, the Maxwell equations of Newtonian mechanics seem to have a gaping *asymmetry*, which arguably *could* be filled by inserting "magnetic charge" and "magnetic current" source terms into these equations.

Thus, despite the author's disbelief in the physical usefulness of magnetic charge, we shall nevertheless consider here the Newtonianly acceptable model of a magnetic dipole as simply being the dual of the electric dipole: two equal and opposite *magnetic charges* on the ends of a stick. The reasons for doing so are fourfold. Firstly, it has been historically common to consider such objects, and no discussion would be complete without such a review. Secondly, it is a simple analysis, being simply obtained from the electric result by means of a trivial duality transformation. Thirdly, we shall begin to appreciate the desirable features of the resulting equations of motion, despite the fact that the model is unphysical. Fourthly, the final equations of motion we shall find, in this chapter, will turn out to be *identical* to the magnetic-charge model—with one, beautiful addition.

Let us therefore immediately take across the results of the previous section, by using the electromagnetic duality transformation

$$\begin{aligned} \boldsymbol{E} &\longrightarrow \boldsymbol{B}, \\ \boldsymbol{B} &\longrightarrow -\boldsymbol{E}, \end{aligned}$$

$$\boldsymbol{d} \longrightarrow \boldsymbol{\mu}. \tag{4.18}$$

We note carefully that *we did not need to use Maxwell's equations at all* in the analysis of the previous section; this is important because, under a duality transformation, the homogeneous and inhomogeneous equations are *interchanged*—and, indeed, need to be *modified* if magnetic charge and current is introduced.

Under the transformations (4.18), we immediately find, for the nonrelativistic analysis,

$$\begin{aligned} P &= (\dot{\boldsymbol{\mu}}\cdot\boldsymbol{B}) + (\boldsymbol{\mu}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{B}), \\ \boldsymbol{F} &= (\boldsymbol{\mu}\cdot\nabla)(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}) - \dot{\boldsymbol{\mu}}\times\boldsymbol{E}, \\ \boldsymbol{N} &= \boldsymbol{\mu}\times(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}). \end{aligned} \tag{4.19}$$

The rigorously relativistic analysis, in the rest frame of the magnetic-charge dipole, likewise yields

$$\begin{aligned} P|_{v=0} &= (\dot{\boldsymbol{\mu}}\cdot\boldsymbol{B}), \\ \boldsymbol{F}|_{v=0} &= (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{B} + (\boldsymbol{\mu}\cdot\dot{\boldsymbol{v}})\boldsymbol{B} - \dot{\boldsymbol{\mu}}\times\boldsymbol{E}, \\ \boldsymbol{N}|_{v=0} &= \boldsymbol{\mu}\times\boldsymbol{B}. \end{aligned} \tag{4.20}$$

The relativistically bootstrapped results for the magnetic-charge dipole are therefore given by

$$\begin{aligned} \dot{p} &= (\mu\cdot\partial)\widetilde{F}\cdot U + [\widetilde{F}\cdot\dot{\mu}], \\ (\dot{S}) &= F\cdot\mu + U(\mu\cdot F\cdot U). \end{aligned} \tag{4.21}$$

4.2.3 The electric-current magnetic dipole

We shall now analyse the most controversial form of magnetic dipole known to physics: that due to the *circulation of electric current*. It is clearly the only physically acceptable magnetic dipole model we have available to us;

but its employment of *moving constituents* makes it a notoriously difficult system to analyse correctly, as we shall shortly find.

The first general observation that the author shall bring to the attention of the reader is the following: The overall power into, force on and torque on an electric-current magnetic dipole *cannot depend on its rate of precession*. Given the results of the previous sections, this may seem surprising; but nevertheless it is a property that is remarkably simple to demonstrate, on quite general grounds. The crucial observation is that the circulating currents of an electric-current magnetic dipole should be *electrically neutral*: they should be formed from the equal and opposite circulation of equal and opposite streams of charge; there should be no *net* electric charge at any position in space. (In other words, they should be pure "electric currents" as an Electrical Engineer would understand the term.) Now, the *precession* of any such loop of current will obviously lead to extra velocities of the circulating charges, but these extra velocities are, for any position in space, *the same for both the positive and the negative charges*. Any power, force or torque coupled to these extra velocities for (say) the *positive* charges would need to be proportional to $q\boldsymbol{v}_{\text{prec}}$ for each charge; but then there would be an equal and opposite coupling $-q\boldsymbol{v}_{\text{prec}}$ to the *negative* charge at that same position in space, because the velocity $\boldsymbol{v}_{\text{prec}}$ is *the same* for both charges. Thus, the power, force and torque due to the precession of the dipole *must vanish at every position in space*, when the positive and negative charge contributions are added together; and hence, *regardless* of what geometrical factors we may wish to insert, the precession can have no effect on these forces whatsoever.

Likewise, when we analyse the system relativistically, according to the *completely rigorous* formulation described in the previous chapter, we *do* find that extra *position-dependent* factors need to be inserted, but these again have no effect, since the forces vanish at every position in space. Thus, *regardless of relativistic effects*, the precession cannot enter into the equations for the forces on the electric-current magnetic dipole.

We furthermore note that *the translational velocity of the current loop as a whole*, $\boldsymbol{v}$, cannot enter into the equations of motion either, in the nonrelativistic analysis, for the same reason as above: the effect of the added velocity to each positive charge, $q\boldsymbol{v}$, is counterbalanced by that added to the negative charge, $-q\boldsymbol{v}$. Of course, we know that a Lorentz boost should transform the quantities involved in the static result, and we *should* obtain *some* terms of order $\boldsymbol{v}$; their omission in the nonrelativistic analysis is of course due to the incorrect treatment of simultaneity in Galilean kinematics, noted in Chapter 2. We shall, at the end of this section, repair this omission by performing the correct "relativistic bootstrap" procedure on the static results.

Now, from the above discussions, we have found that neither precession nor translational velocity of the electric-current magnetic dipole will have any effect on the forces obtained. We are therefore left with considering the simple case of a *static, non-precessing* current loop. Let us consider some particular geometry of an electric-current magnetic dipole that is easy to analyse, in the same way that we analysed the "charges on a stick" model for an electric dipole. We must note that the *particular* geometry chosen for the electric-current magnetic dipole is *arbitrary*: the final results will be found to be the same regardless of the choice one makes; all that is relevant is that the dipole moment *does* arise through the circulation of electric current.

We shall therefore consider a simple *circular, planar loop of electric current*, of radius ε, with a line of positive electric charge circulating in one direction, and an equal and opposite line of negative electric charge circulating in the other direction. It will simplify the analysis to construct a set of Cartesian coördinates, with the centre of the circular loop at the origin of coördinates, and the loop itself lying in the x–y plane; we use the symbols $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to denote the unit vectors in the x, y and z directions respectively. We denote the angle between the x-axis and the radius vector to any given position on the circular loop by θ, with the usual convention that $\theta = +\pi/2$ when this radius vector points along the positive y-axis. The position of that point of the loop at angle θ to the x-axis is then given by

$$\boldsymbol{z}(\theta) = \varepsilon\Big\{\boldsymbol{i}\,\cos\theta + \boldsymbol{j}\sin\theta\Big\}. \tag{4.22}$$

Clearly, the direction of the magnetic moment $\boldsymbol{\mu}$ of the current loop will lie along the z-axis. We choose to make it point in the *positive* z-direction:

$$\boldsymbol{\mu} \equiv \mu\boldsymbol{k}. \tag{4.23}$$

This then requires the (Engineer's) electric current I in the loop to flow in the direction of positive θ, *i.e.*, counter-clockwise as viewed from a position "above" the loop ($z > 0$). The definition of the magnitude μ of a planar electric-current magnetic dipole is simply the product of the area of the loop by the Engineer's current I flowing around it [113]; in our case,

$$\mu \equiv \pi\varepsilon^2 I. \tag{4.24}$$

Now, we are here forming the Engineer's current I from equal and opposite lineal streams of positive and negative charge, around the loop: the charge densities cancel, and their current densities add. Thus, half of I will be due to the integral of the $q\boldsymbol{v}$ contributions of the positive charges, and half will be due to the same integral of the contributions of the negative charges. We denote the total number of positive charges circulating in the loop as n; the number of negative charges circulating in the loop is thus also n. The number of charges n will be taken to infinity at the end of the calculations, and the product of the other quantities taken to zero to compensate, to provide a "continuous" stream of charge. Each individual positive or negative charge is denoted $+q$ or $-q$ respectively. We denote the *speed* of each charge in its "orbital" motion around the loop by v_{orb}. The Engineer's current I is then simply given by

$$I = 2\,\frac{nqv_{\text{orb}}}{2\pi\varepsilon}, \tag{4.25}$$

where the factor 2 outside the front is due to the presence of both positive *and* negative charges in circulation; the factor $2\pi\varepsilon$ is the circumference of the loop. We can now substitute (4.25) into (4.24), to eliminate the Engineer's current I altogether, in favour of the quantities n, q, v_{orb} and ε:

$$\mu = nq\varepsilon v_{\text{orb}}. \tag{4.26}$$

Now, since the positive charges are simply moving around the loop at speed v_{orb}, in the direction of increasing θ, we then have

$$\boldsymbol{v}_+(\theta) = v_{\text{orb}}\Big\{-\boldsymbol{i}\,\sin\theta + \boldsymbol{j}\cos\theta\Big\}, \tag{4.27}$$

where by $\boldsymbol{v}_+(\theta)$ we denote the velocity of each positive charge, when at the positional angle θ in the loop. The negative charge at any position of the loop is moving in the opposite direction to the positive charge at that same position, and thus

$$\boldsymbol{v}_-(\theta) = -\boldsymbol{v}_+(\theta). \tag{4.28}$$

We again expand the electric and magnetic fields as Taylor series around the centre of the loop:

$$\begin{aligned}\boldsymbol{E}(\theta) &= \boldsymbol{E} + \varepsilon\Big\{\cos\theta\,\partial_x + \sin\theta\,\partial_y\Big\}\boldsymbol{E} + \mathrm{O}(\varepsilon^2),\\ \boldsymbol{B}(\theta) &= \boldsymbol{B} + \varepsilon\Big\{\cos\theta\,\partial_x + \sin\theta\,\partial_y\Big\}\boldsymbol{B} + \mathrm{O}(\varepsilon^2).\end{aligned} \tag{4.29}$$

Finally, we note that, in the limit that $n \to \infty$, it makes sense to talk of the differential number of charges dn in the differential angle $d\theta$; clearly,

$$dn = \frac{n}{2\pi}\,d\theta. \tag{4.30}$$

We now have all the ingredients necessary for an analysis of our planar current loop. We start with the computation of the power into the loop. This is given by

$$P = \int dn\,\Big\{P_+(n) + P_-(n)\Big\},$$

where we are taking n as an integration variable, again anticipating the continuum limit $n \to \infty$. We can convert this integral in n to one in θ by using (4.30):

$$P = \frac{n}{2\pi}\int_0^{2\pi} d\theta\,\Big\{P_+(\theta) + P_-(\theta)\Big\}. \tag{4.31}$$

Now, the power into each positive charge is given by the Lorentz result

$$P_+(\theta) = q\boldsymbol{v}_+(\theta)\cdot\boldsymbol{E}(\theta),$$

and the power into each negative charge is similiarly given by

$$P_-(\theta) = -q\boldsymbol{v}_-(\theta)\cdot\boldsymbol{E}(\theta),$$

Using (4.28), we therefore see that

$$P(\theta) = 2q\boldsymbol{v}_+(\theta)\cdot\boldsymbol{E}(\theta), \tag{4.32}$$

where

$$P(\theta) \equiv P_+(\theta) + P_-(\theta).$$

Substituting (4.32) into (4.31), using (4.27) and (4.29), and integrating, we thus find

$$P = nq\varepsilon v_{\text{orb}}\Big\{\partial_x E_y - \partial_y E_x\Big\} + \mathrm{O}(\varepsilon^2).$$

Noting the identities (4.23) and (4.26), and taking the point limit, we therefore find

$$P = \boldsymbol{\mu}\cdot\nabla\times\boldsymbol{E}. \tag{4.33}$$

We now use one of the homogeneous Maxwell equations,

$$\nabla\times\boldsymbol{E} + \partial_t\boldsymbol{B} = \boldsymbol{0},$$

to obtain our final result

$$P = -\boldsymbol{\mu}\cdot\partial_t\boldsymbol{B}. \tag{4.34}$$

The result (4.34) is highly undesirable. It is also untuitively obvious. It tells us that if there is a *changing magnetic field* through the current loop, then energy will be transferred into the loop by induction. But this occurs for a *stationary* loop. If we tried to apply this equation to an *electron*, say, we would then (relativistically speaking) find that its *mass could be changed*, simply by ramping up a magnetic field, at the position of the electron, in the direction of its magnetic moment. This is clearly unphysical. This is why the current loop equations have not been popular this century. This will be repaired shortly.

We now turn to the question of the *force* on the electric-current magnetic dipole. In this nonrelativistic analysis, we again simply sum the forces on all of the circulating constituent charges:

$$\boldsymbol{F} = \frac{n}{2\pi}\int_0^{2\pi} d\theta \left\{ \boldsymbol{F}_+(\theta) + \boldsymbol{F}_-(\theta) \right\}. \tag{4.35}$$

The Lorentz force on each positive charge gives

$$\boldsymbol{F}_+(\theta) = q\Big\{ \boldsymbol{E}(\theta) + \boldsymbol{v}_+(\theta) \times \boldsymbol{B}(\theta) \Big\},$$

and, likewise, for each negative charge,

$$\boldsymbol{F}_-(\theta) = -q\Big\{ \boldsymbol{E}(\theta) + \boldsymbol{v}_-(\theta) \times \boldsymbol{B}(\theta) \Big\}.$$

Again using (4.28), we see that

$$\boldsymbol{F}(\theta) = 2q\boldsymbol{v}_+(\theta) \times \boldsymbol{B}(\theta), \tag{4.36}$$

where

$$\boldsymbol{F}(\theta) \equiv \boldsymbol{F}_+(\theta) + \boldsymbol{F}_-(\theta).$$

Substituting (4.36) into (4.35), again using (4.27) and (4.29), and integrating, we thus find

$$\boldsymbol{F} = nq\varepsilon v_{\rm orb}\Big\{ \boldsymbol{i}\,\partial_x B_z + \boldsymbol{j}\partial_y B_z - \boldsymbol{k}(\partial_x B_x + \partial_y B_y) \Big\} + \mathrm{O}(\varepsilon^2). \tag{4.37}$$

We now need to use the other homogeneous Maxwell equation,

$$\nabla \cdot \boldsymbol{B} = 0;$$

in component form, this reads

$$\partial_x B_x + \partial_y B_y + \partial_z B_z = 0;$$

or, on reärranging the terms,

$$-(\partial_x B_x + \partial_y B_y) = \partial_z B_z. \tag{4.38}$$

But the left-hand side of (4.38) is simply the parenthesised expression in (4.37); hence, we find

$$\boldsymbol{F} = nq\varepsilon v_{\text{orb}} \Big\{ \boldsymbol{i}\,\partial_x B_z + \boldsymbol{j}\partial_y B_z + \boldsymbol{k}\partial_z B_z \Big\} + \mathrm{O}(\varepsilon^2).$$

But the expression in braces is simply ∇B_z; hence,

$$\boldsymbol{F} = nq\varepsilon v_{\text{orb}} \nabla B_z.$$

Again noting the relations (4.23) and (4.26), and taking the point limit, we therefore find

$$\boldsymbol{F} = \nabla(\boldsymbol{\mu}\cdot\boldsymbol{B}). \tag{4.39}$$

The expression (4.39) is the force on an electric-current magnetic dipole. It is the controversial "textbook force" on a magnetic dipole. For example, we find the following in Jackson [113, Sec. 5.7]:

> This can be written vectorially as
>
> $$\boldsymbol{F} = (\boldsymbol{m}\times\nabla)\times\boldsymbol{B} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}) - \boldsymbol{m}(\nabla\cdot\boldsymbol{B}).$$
>
> Since $\nabla\cdot\boldsymbol{B} = 0$ generally, the lowest order force on a localized current distribution in an external magnetic field $\boldsymbol{B}$ is
>
> $$\boldsymbol{F} = \nabla(\boldsymbol{m}\cdot\boldsymbol{B}).$$
>
> This result holds even for time-varying external fields.

These statements are absolutely correct. (By "lowest-order" Jackson is referring to the keeping of only the first term in the Taylor expansion (4.29) of the magnetic field; this is rigorously true in the point limit.) Equation (4.39) *is* the force on the electric-current magnetic dipole. But there is again a twist; we shall return to this shortly.

We now turn to the question of the *torque* on an electric-current magnetic dipole. Fortunately, this problem is not subject to any controversy whatsoever. In this nonrelativistic analysis, we simply sum the "moment-arm" torques on all of the circulating constituent charges:

$$\boldsymbol{N} = \frac{n}{2\pi}\int_0^{2\pi} d\theta \left\{ \boldsymbol{N}_+(\theta) + \boldsymbol{N}_-(\theta) \right\}. \tag{4.40}$$

Since each torque is only dependent on the *force* at that position θ, we can immediately make use of the *sum* of the torques on the positive and negative charges at that position:

$$\boldsymbol{N}(\theta) \equiv \boldsymbol{N}_+(\theta) + \boldsymbol{N}_-(\theta) \equiv \boldsymbol{z}(\theta) \times \boldsymbol{F}(\theta).$$

From (4.36), we find

$$\boldsymbol{N}(\theta) = 2q\boldsymbol{z}(\theta) \times \Big(\boldsymbol{v}_+(\theta) \times \boldsymbol{B}(\theta) \Big);$$

using the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) \equiv (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c},$$

and noting that

$$\boldsymbol{z}(\theta) \cdot \boldsymbol{v}_+(\theta) \equiv 0$$

(since the charge is moving in circular motion, *i.e.*, with velocity perpendicular to its radius vector), we thus find

$$\boldsymbol{N}(\theta) = 2q\Big(\boldsymbol{z}(\theta) \cdot \boldsymbol{B}(\theta) \Big) \boldsymbol{v}_+(\theta). \tag{4.41}$$

Substituting (4.41) into (4.40), using (4.22), (4.27) and (4.29), and integrating, we find

$$\boldsymbol{N} = nq\varepsilon v_{\text{orb}} \Big\{ -\boldsymbol{i}\, B_y + \boldsymbol{j} B_x \Big\} + \mathrm{O}(\varepsilon^2). \tag{4.42}$$

Again noting the relations (4.23) and (4.26), and taking the point limit, we therefore find

$$\boldsymbol{N} = \boldsymbol{\mu} \times \boldsymbol{B}. \tag{4.43}$$

Equation (4.43) is the (completely uncontroversial) expression for the torque on an electric-current magnetic dipole. To see *why* it is uncontroversial, we only need note that—unlike the power and force expressions—this result (4.43) is the *same* as the result (4.19) found for the magnetic-charge dipole in the previous section.

In fact, Hraskó [108] has shown why one *must* obtain this torque law for *any* object generating a magnetic dipole field, *regardless* of the nature of the object. He likewise showed why the Lorentz force *must* be obtained for any electric monopole. His proofs considered the mechanical momentum contained in the electromagnetic field surrounding the position of the charge or magnetic dipole, using essentially the same concept of "conservation of mechanical four-momentum" as the derivation in Section 2.3.8 of the Lorentz force. In fact, his proofs are much more satisfactory: he essentially uses the Dirac [68] method of considering the flow of mechanical four-momentum (via the mechanical stress-energy tensor) through the surface of a small volume surrounding the particle; this has the great theoretical advantage that the proof is now *manifestly local*, and independent of retardation effects, *etc.* Now, Hraskó found that the contributions to the integrals of the *interior* of the small volume surrounding the point particle *vanish*, for the monopole force and dipole torque, when the volume is shrunk to zero. However, for the *power and force* on a dipole, this integral over the small volume surrounding the particle *does not vanish*: it is, in fact, finite; and hence the internal field differences between the magnetic-charge dipole and electric-current magnetic dipole can (and do) lead to different power and force expressions. (The reason that these integrals do not vanish as with the monopole case is that the dipole field contains one extra power of $1/r$ over that of the monopole field;

this brings a vanishing integrand r-dependence up to a finite r-dependence. The dipole *torque* avoids this problem because *the angular momentum has the vector* $\boldsymbol{r}$ *explicitly crossed into its definition*, and hence the integrand in question only diverges to the same order as the corresponding integrand for the force on a monopole field, *i.e.*, it vanishes over the small volume surrounding the particle.)

Let us now make the above analyses relativistically rigorous, by again employing the rigid body formalism described in Chapter 3. We need to add in the effects of the accelerative redshift factor,

$$\lambda(\boldsymbol{r}) \equiv 1 + (\boldsymbol{r} \cdot \dot{\boldsymbol{v}}).$$

At any position in our circular loop, we use (4.22) to find

$$\lambda(\theta) = 1 + \varepsilon \Big\{ \dot{v}_x \cos\theta + \dot{v}_y \sin\theta \Big\}. \tag{4.44}$$

If one examines the expressions carefully, one finds that the redshift correction terms in (4.44), of order ε, will not contribute to the torque integral, because that integral already has $\boldsymbol{z}(\theta)$ crossed into it, which is itself of order ε. On the other hand, we do find contributions to the *power* and *force* integrals. For the former, one finds

$$P_{\text{redshift}} = \boldsymbol{\mu} \cdot \dot{\boldsymbol{v}} \times \boldsymbol{E}.$$

Now, if one compares this to our original result (4.33), and then looks back at the electric dipole redshift force found in Section 4.2.1, one finds that we seem to be establishing a "redshift prescription",

$$\nabla \longrightarrow \nabla + \dot{\boldsymbol{v}}. \tag{4.45}$$

Again, since this effect was only discovered by the author a few days before the printing of this thesis, only a small amount of contemplation has followed its discovery. Mathematically, the above prescription of course arises because the quantity

$$(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})$$

has the same position dependence as the quantity

$$(\boldsymbol{r}\cdot\nabla)$$

in the Taylor expansion of the fields. However, the author does not yet have any intuitively simple explanation of why the prescription (4.45) is *physically* correct. We leave this as an exercise for the reader.

We now turn to the redshift correction to the force expression. As with the above, the detailed analysis again finds that this extra contribution may be obtained by means of the redshift prescription (4.45):

$$\boldsymbol{F}_{\text{redshift}} = \dot{\boldsymbol{v}}(\boldsymbol{\mu}\cdot\boldsymbol{B}).$$

4.2.4 Mechanical momentum of the current loop

We now ask the following loaded question: What is the mechanical momentum of a stationary electric-current magnetic dipole?

Before we answer this question, let us first explain its importance. In Chapter 2, we continually (and somewhat pedantically) reminded the reader that *the force on a particle is the time rate of change of its mechanical momentum*:

$$\boldsymbol{F} \equiv d_t\boldsymbol{p}.$$

This is the *only* definition of "force" permissible. If you were to inscribe the laws of physics on stone tablets, this would go on the first one.

Now, a point mass, possessing no other characteristics at all, has the following kinematical relationship between its velocity and its mechanical momentum:

$$\boldsymbol{p} \equiv m\gamma\boldsymbol{v}. \tag{4.46}$$

This would seem to imply, by the use of the relativistic centre of energy theorem, that, for a *system* of particles, one should have

$$\boldsymbol{P} \equiv M\Gamma\boldsymbol{V}, \tag{4.47}$$

where $\boldsymbol{P}$ is the mechanical momentum of the system as a whole (the sum of the mechanical momenta of its constituents), M is the mechanical rest-energy of the system, $\boldsymbol{V}$ the velocity of the centre of energy of the system, and Γ is the gamma factor corresponding to $\boldsymbol{V}$.

The problem is that this use of the centre of energy theorem *relies on the fact that the system is isolated*. For an isolated system, (4.47) is indeed true. But if the system is *not* isolated, it *need not* remain true. For an electric charge, it is still true. For an electric dipole, it therefore also remains true. For a magnetic-charge dipole, it still remains true. . . .

The author has stopped his list one short of the end. The problem with the *electric-current magnetic dipole* is not that its constituents are electric charges, but that *these constituents are free to move around their constraining "tube"*: they retain one degree of freedom. When the electric-current magnetic dipole is *isolated*, it of course still obeys the centre of energy theorem result (4.47). But when it is *in the electromagnetic field of other electric sources in the Universe*, the result (4.47) fails. This was first recognised by Penfield and Haus [170, 171, 102]. The author shall now outline their argument, using the formalism and notation of the previous section.

The essential agent of causation in the Penfield–Haus effect is the presence of *an external electric field* $\boldsymbol{E}$ *whose direction is in the plane of the current loop* (*i.e.*, perpendicular to the magnetic moment $\boldsymbol{\mu}$), at the position of the electric-current magnetic dipole. The spatial variation of $\boldsymbol{E}$ over the loop may be neglected; the corresponding corrections vanish in the point limit. Since our current loop formalism of the previous section is circularly symmetric in the plane of the loop, we may choose an arbitrary direction, in the x–y plane, for the direction of the external electric field $\boldsymbol{E}$: let us choose the positive-y

direction:

$$\boldsymbol{E} \equiv \boldsymbol{j} E_y.$$

Now, let us analyse the motion of one of the constituent positive charges in our loop, as it makes its journey around one complete orbit. For simplicity, we start the charge (at time $t = 0$) at the position $\theta = 0$. Now, as the charge moves counter-clockwise around the loop with speed v_{orb}, it feels the effect of the electric field $\boldsymbol{E}$: namely, the Coulomb force $q\boldsymbol{E}$, which in our configuration is in the y-direction:

$$\boldsymbol{F}_E = \boldsymbol{j} q E_y.$$

When the charge is at position $\theta = \pm\pi/2$, this force is wholly perpendicular to the motion of the charge, and is absorbed by the "rigid walls" of the constraining tube. But when the charge is at the position $\theta = 0$ or $\theta = \pi$, it feels a *force parallel or antiparallel to the direction of its motion*. At a general position in the loop, the force along its motion is given by

$$d_t p_\theta = q E_y \cos\theta, \tag{4.48}$$

where we are here considering only the mechanical momentum in the theta-direction, p_θ (since the charge is constrained to move in only that direction); the factor $\cos\theta$ is simply $\hat{\boldsymbol{\theta}} \cdot \boldsymbol{j}$. Let us refer to the *additional* mechanical momentum given to the charge, over and above that due to its designed orbital motion, as $\varDelta p_\theta(t)$, where

$$\varDelta p_\theta(0) \equiv 0. \tag{4.49}$$

This extra mechanical momentum given to the charge by the electric field will cause it to accelerate and decelerate as it circulates.

An essential problem now arises: the *positive* charges will be moving fastest the position $\theta = \pi/2$ (*i.e.*, when they are as far in the direction of the field $\boldsymbol{E}$ as they can be), whereas the *negative* charges will be moving fastest at the position $\theta = -\pi/2$. This implies that *the net charge densities will no longer balance at each point of the loop*; in particular, we will find that it now possesses an *induced electric dipole moment*; the integrity of its magnetic dipole moment may also be lost. Essentially, we are feeling the ramifications of letting our constituent charges retain a degree of freedom: they are now "sympathetic" to the external fields.

Now, if the physical situation that one would like to consider *is*, indeed, made up of a number of real electric charges in circulation, with some given velocities, then this induced electric dipole moment is a reality, and needs to be considered in the equations of motion of the system. On the other hand, if one is actually trying to construct an appropriate model of a *fixed* magnetic dipole moment, *without* an induced electric dipole moment (as we are for the purposes of this thesis, namely, a model that is applicable to the fixed intrinsic moments of spin-half particles), then this is not acceptable. We must therefore take steps to ensure that an appropriate limit is taken that eliminates this unwanted induced moment.

Now, if one considers the situation from first principles, one can see that if the initial speeds of the circulating charges are *small*, then the charges will essentially act as if they are free, and the positive charges will simply tend to crowd together at one end of the loop, and the negative charges at the other end, ultimately only constrained by their mutual repulsion. This is of course what we *do not* want to happen. So let us, following Penfield and Haus [171], look at the *opposite* limit: that of *ultra-relativistic* circulating charges. We then of course know that the extra mechanical momentum absorbed and relinquished during each orbit will lead to only *small* changes in speed, due to the relativistic relationship between mechanical momentum and three-velocity. Moreover, the *time* that each charge spends in any single orbit will also be reduced, down to the limiting value (for given ε)

$$\tau_{\min} = 2\pi\varepsilon$$

(*i.e.*, $2\pi\varepsilon/v_{\rm orb}$ with $v = 1$, the speed of light), and so the effects of the electric field on its orbital motion will be minimised further, since the impulse given to the charge on its "downhill leg" is of order $qE\tau$. Thus, we find that we in fact need to take the constituent charges to be in the *ultra-relativistic limit* for our current purposes.

Let us, however, retain the quantity $v_{\rm orb}$ for the remainder of this derivation, and only set it to unity at the end; this will show us *quantitatively* why the ultra-relativistic limit is necessary. Now, the angular position of our chosen charge, as a function of time, may be obtained quite simply from its orbital motion; if we assume its speed to be *approximately constant* throughout its orbit, then

$$\theta(t) = \frac{v_{\rm orb} t}{\varepsilon}, \tag{4.50}$$

since it makes one complete orbit in a time period

$$\tau_{\rm orb} = \frac{2\pi\varepsilon}{v_{\rm orb}},$$

and hence

$$\omega_{\rm orb} \equiv \frac{2\pi}{\tau_{\rm orb}} \equiv \frac{v_{\rm orb}}{\varepsilon}.$$

Using (4.50) in (4.48), we therefore find a differential equation for the zeroth-order contribution to $\mathit{\Delta} p_\theta(t)$:

$$d_t \mathit{\Delta} p_\theta = qE_y \cos\left(\frac{v_{\rm orb} t}{\varepsilon}\right),$$

with the initial condition (4.49). This is trivially integrated:

$$\mathit{\Delta} p_\theta = \frac{q\varepsilon E_y}{v_{\rm orb}} \sin\left(\frac{v_{\rm orb} t}{\varepsilon}\right).$$

We now reïnstate the full vectorial nature of $\mathit{\Delta}\boldsymbol{p}_\theta$, by using the identity

$$\hat{\boldsymbol{\theta}} \equiv -\boldsymbol{i} \sin\theta + \boldsymbol{j} \cos\theta;$$

thus, using (4.50), we find

$$\Delta\boldsymbol{p}_+(t) = \frac{q\varepsilon E_y}{v_{\text{orb}}}\sin\left(\frac{v_{\text{orb}}t}{\varepsilon}\right)\left\{-\boldsymbol{i}\sin\left(\frac{v_{\text{orb}}t}{\varepsilon}\right) + \boldsymbol{j}\cos\left(\frac{v_{\text{orb}}t}{\varepsilon}\right)\right\}. \tag{4.51}$$

Now, if we average this expression over one orbit of the charge, and multiply by the number of positive charges in the loop, we find

$$\Delta\boldsymbol{p}_+ = \frac{n}{\tau_{\text{orb}}}\int_0^{\tau_{\text{orb}}} dt\,\Delta\boldsymbol{p}_+(t) = -\boldsymbol{i}\,\frac{q\varepsilon E_y}{2v_{\text{orb}}}.$$

If we now look at performing the above analysis for a *negative* charge in the loop, we must simply change $+q$ to $-q$, and reverse the sign of $\theta(t)$ in (4.50): this gives us

$$\Delta\boldsymbol{p}_-(t) = -\frac{q\varepsilon E_y}{v_{\text{orb}}}\sin\left(\frac{v_{\text{orb}}t}{\varepsilon}\right)\left\{+\boldsymbol{i}\sin\left(\frac{v_{\text{orb}}t}{\varepsilon}\right) + \boldsymbol{j}\cos\left(\frac{v_{\text{orb}}t}{\varepsilon}\right)\right\}, \tag{4.52}$$

where the negative sign out the front is that due to $q \rightarrow -q$, and the first term in braces has changed sign because $\sin(-\theta) \equiv -\sin(\theta)$. The integral of (4.51), over all of the negative charges, thus also gives

$$\Delta\boldsymbol{p}_- = \frac{n}{\tau_{\text{orb}}}\int_0^{\tau_{\text{orb}}} dt\,\Delta\boldsymbol{p}_-(t) = -\boldsymbol{i}\,\frac{q\varepsilon E_y}{2v_{\text{orb}}}.$$

Thus, adding the contributions of the negative and positive charges together, we find

$$\Delta\boldsymbol{p} \equiv \Delta\boldsymbol{p}_+ + \Delta\boldsymbol{p}_- = -\boldsymbol{i}\,\frac{q\varepsilon E_y}{v_{\text{orb}}}.$$

Using (4.26), and noting that $\boldsymbol{\mu}\times\boldsymbol{E}$ is, for our configuration, in the negative-x direction, we therefore find

$$\Delta\boldsymbol{p} = \frac{\boldsymbol{\mu}\times\boldsymbol{E}}{v_{\text{orb}}^2}. \tag{4.53}$$

Now, this result (4.53) seems to imply that, in the *non*relativistic limit of the motion of the constituent charges, their net mechanical momentum would *diverge*. This is, of course, not the case: if the charges are not moving with a sufficient amount of inertia, the cyclical changes to their velocity during each orbit, due to the force of the electric field, may be as large as the designed orbital velocity: in fact, in the nonrelativistic limit it completely *dominates* their motion; hence, the assumption of roughly constant orbital speed, underlying the expression (4.50), is invalidated in the nonrelativistic limit, and the expression (4.53) cannot be applied to that regime.

On the other hand, we see that, in the *ultra*-relativistic limit—where the speed of each charge rigorously remains at practically the speed of light,—the result (4.53) gives

$$\varDelta \boldsymbol{p} = \boldsymbol{\mu} \times \boldsymbol{E}. \tag{4.54}$$

In this limit, the magnitude of the magnetic dipole moment is constant, the induced electric dipole moment vanishes, and there is a *model-independent contribution*, (4.54), to the mechanical momentum of the electric-current magnetic dipole.

Now, if we are considering the ultra-relativistic current loop as a model for a fixed magnetic dipole moment, then we clearly need to take account of the extra mechanical momentum (4.54) when computing the equations of motion for the loop as a whole. If the quantity $\boldsymbol{\mu} \times \boldsymbol{E}$ is *changing*—for whatever reason—then the ultra-relativistic constituent charges will effectively absorb or relinquish some of the extra mechanical momentum (4.54) that they are in possession of. If m is the mechanical rest-energy of the loop as a whole, and $\boldsymbol{v}$ is its velocity, we thus see that

$$\boldsymbol{F} \equiv d_t \boldsymbol{p} = d_t(m\gamma \boldsymbol{v} + \varDelta \boldsymbol{p}).$$

In other words, we find

$$d_t(m\gamma \boldsymbol{v}) = \boldsymbol{F} - d_t \varDelta \boldsymbol{p}. \tag{4.55}$$

Inserting the expression for the force on an arbitrary stationary electric-current magnetic dipole, found in the previous section:

$$\boldsymbol{F} = \nabla(\boldsymbol{\mu} \cdot \boldsymbol{B})$$

(we shall reïnsert the new redshift force shortly), and the expression (4.54) for the mechanical momentum $\Delta \boldsymbol{p}$ for the stationary electric-current magnetic dipole with ultra-relativistic constituents, into (4.55), we thus find the equation of motion

$$\begin{aligned} d_t(m\gamma\boldsymbol{v}) &= \nabla(\boldsymbol{\mu}\cdot\boldsymbol{B}) - d_t(\boldsymbol{\mu}\times\boldsymbol{E}) \\ &= (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{B} + \boldsymbol{\mu}\times(\nabla\times\boldsymbol{B}) - \dot{\boldsymbol{\mu}}\times\boldsymbol{E} - \boldsymbol{\mu}\times d_t\boldsymbol{E}, \end{aligned}$$

by using a three-vector identity on the first part. Now, the loop we are considering in this section is stationary, so the convective derivative

$$d_t\boldsymbol{E} \equiv \partial_t\boldsymbol{E} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{E}$$

is in this case simply equal to $\partial_t\boldsymbol{E}$. Using Maxwell's equations for $\nabla\times\boldsymbol{B}$, we thus find

$$d_t(m\gamma\boldsymbol{v}) = (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{B} + \boldsymbol{\mu}\times(\partial_t\boldsymbol{E} + \boldsymbol{J}) - \dot{\boldsymbol{\mu}}\times\boldsymbol{E} - \boldsymbol{\mu}\times\partial_t\boldsymbol{E}.$$

Thus, for the ultra-relativistic current-loop model of a fixed magnetic dipole, we find

$$d_t(m\gamma\boldsymbol{v}) = (\boldsymbol{\mu}\cdot\nabla)\boldsymbol{B} - \dot{\boldsymbol{\mu}}\times\boldsymbol{E} + \boldsymbol{\mu}\times\boldsymbol{J}. \tag{4.56}$$

Apart from the last term, this is *identical to the equation of motion for the magnetic-charge dipole*. The last term in (4.56) is a *contact force* between the magnetic dipole $\boldsymbol{\mu}$ and any external current $\boldsymbol{J}$ that is generating a magnetic field. It is beautifully appropriate that the magnetic-charge and ultra-relativistic current-loop dipoles only behave differently *if one probes the internal structure of the magnetic dipole*, by making it collide with an external current $\boldsymbol{J}$; if, on the other hand, we keep our external currents $\boldsymbol{J}$ *away* from the position of the dipole—and hence our external currents only "see" the *external* magnetic dipole field, identical for either model—then we find that the two types of dipole also *move* identically. One could not hope for a more æsthetically pleasing result.

The fact that the Penfield–Haus effect returns us to practically the same equation of motion as the magnetic-charge dipole means that the undesirable properties of the current-loop equations of motion are no longer applicable; we instead return to the manifestly satisfactory properties of the electric dipole case. (The extra contact force can be easily shown to not lead to any change in the mass of the particle either.) Thus, since the mass m is a constant, we can write down the covariant equation of motion, in terms of the four-velocity U, immediately:

$$m\dot{U} = (\mu\cdot\partial)\widetilde{F}\cdot U + [\widetilde{F}\cdot\dot{\mu}] + \mu\times J\times U; \tag{4.57}$$

the final term is the covariant generalisation of the contact force $\boldsymbol{\mu}\times\boldsymbol{J}$, and the partial derivative $[\dot{\mu}]$ encompasses the new redshift force, as with the electric dipole. The covariant spin equation of motion is of course just the Bargmann–Michel–Telegdi equation:

$$(\dot{S}) = F\cdot\mu + (\mu\cdot F\cdot U)U. \tag{4.58}$$

Equations (4.57) and (4.58) are essentially the final classical equations of motion the author shall present for a fixed pointlike magnetic dipole (ignoring radiation reaction, which will be considered in Chapter 6).

4.2.5 Literature on the current loop force law

If the reader finds the Penfield–Haus effect interesting, they will find some of the literature on the subject simply fascinating. At essentially the same time as Penfield and Haus were doing their work on the subject, Shockley and James [188] constructed a beautiful *gedanken* experiment that showed that something was definitely amiss with the standard textbook force on a current loop. They essentially placed a stationary charged particle at some distance from two counter-rotating charged disks (their electric-current magnetic dipole), and then introduced a very small frictional force between the disks. The magnetic dipole moment then decays slowly to zero, due to this frictional force. The crucial observation is that the time-changing magnetic dipole force induces, by Maxwell's equations, an *electric* field in the space around the dipole. One can show that this electric field is given by

$$\boldsymbol{E}(t) = \frac{\boldsymbol{n}\times\dot{\boldsymbol{\mu}}}{4\pi r^2}.$$

(This can, incidentally, also be obtained from the author's retarded field expressions of Chapter 5, even though the author only considers $\dot{\boldsymbol{\mu}}$'s for which μ stays constant; it appears this result is quite general.) This electric field acts on the charge, and gives it a mechanical momentum impulse. Because the electric field depends on the rate of change of the magnetic moment, but the impulse integrates this force up again, the net impulse is independent of how slowly one lets the magnetic moment decay.

Shockley and James then point out that there is no apparent counter-balancing force on the magnetic dipole! By taking the limit in which the charge's m/q ratio is made to approach infinity, the velocity attained by the charge due to the imparted impulse can be made as arbitrarily small as one likes, and hence there is no substantial magnetic field induced by the charge that could act on the magnetic dipole. This may be taken to imply that the mechanical momentum of the total system is not conserved.

Shockley and James suggested that a mechanical momentum excess contained in the *electromagnetic field* is a "hidden momentum" of the magnetic dipole; they then used this essentially as the $\Delta\boldsymbol{p}$ of the Penfield–Haus effect. However, *this is manifestly incorrect*, as the discussions of Chapter 2 show: the mechanical field momentum excess has *already been counted* in the derivation of the Lorentz force law, which in turn yields the "textbook" current loop force law. Thus, to add this excess field mechanical momentum *again* would be to count it twice.

In fact, if one considers the question carefully, one finds that in fact there are *three* aspects of the physical situation that all have a mechanical momentum of $\pm\boldsymbol{\mu}\times\boldsymbol{E}$. Firstly, one can show that the mechanical field momentum excess for an *electric* dipole in a *magnetic* field is in fact given by

$$\boldsymbol{p}_{\text{excess}} = -\boldsymbol{d}\times\boldsymbol{B}; \tag{4.59}$$

we will shortly find that this same result can be obtained from the Lagrangian description of the electric dipole, as the difference between the canonical and mechanical momentum of the particle, in the same way that we found the quantity $q\boldsymbol{A}$ for an electric charge. One might therefore expect the *dual* of the result (4.59), namely,

$$\boldsymbol{p}_{\text{excess}} = \boldsymbol{\mu}\times\boldsymbol{E}, \tag{4.60}$$

to be applicable to the magnetic dipole. But this ignores the fact that there is *an extra delta-function field* $\boldsymbol{B}_M \equiv \boldsymbol{\mu}\,\delta(\boldsymbol{r})$ *at the position of the magnetic dipole*, over and above the dual of the electric dipole field (*see* Chapter 5); thus, we get an *extra* field contribution for the current loop, of value

$$\boldsymbol{p}_{\text{excess}} = \int d^3r\,\boldsymbol{E}\times\boldsymbol{\mu}\,\delta(\boldsymbol{r}) \equiv -\boldsymbol{\mu}\times\boldsymbol{E}.$$

But this mechanical field momentum contribution *cancels* that of (4.60). In other words, *the current loop has no net mechanical field momentum excess at all*; this is again verified by the Lagrangian analysis, which finds *no* difference between the canonical and mechanical momentum for the current loop. (One can intuitively understand this result by recalling that in an *electric* dipole, Faraday lines of electric field have beginnings and ends; but in a *magnetic* dipole they do not: they are closed paths, since there are no magnetic charges.)

Finally, we of course have the third contribution to the consideration of the current loop, of value $+\boldsymbol{\mu}\times\boldsymbol{E}$: the Penfield–Haus mechanical momentum of the constituents themselves. Unlike the mechanical field momentum excesses, this contribution, being due not to the fields but to the motion of the particles themselves, *does* need to be taken into account over and above the Lorentz force law, in the equation of motion for the particle, in the way shown in the previous section.

Unfortunately, the incorrect "field mechanical momentum" explanation of Shockley and James of the Penfield–Haus effect seems to have been accepted *carte blanche*, in more recent years, by some authors: by Aharonov and Casher in their paper on the effect that bears their name [7]; and, even more recently, in a paper by Aharonov, Pearle and Vaidman [8] that attempts to make the argument more rigorous. The latter is most unfortunate, because the main message of the Aharonov–Pearle–Vaidman paper is most definitely *correct* (the establishment of the correct law of motion for the current loop; the fact that this leads to no force in the Aharonov–Casher effect); but their use of the field mechanical momentum is badly described, and physically *incorrect*. They also state that either the Shockley and James argument *or* the Penfield and Haus argument may be alternatively chosen to explain the Penfield–Haus effect: this is not so, of course, since the two arguments deal with *different* physical aspects of the system, and so if they were *both* to be correct, it would imply an effect twice as large!

Finally, if one wishes to examine a masterpiece of detective work, one must look at the 1968 paper by Coleman and Van Vleck [55] on the Shockley and James paradox. This paper establishes, in a most careful and rigorous way, that there *is indeed* some amount of mechanical momentum that is missing. They then employ the Darwin Lagrangian to show explicitly how the interactions between the charges in the current loop and the external charge arise. They establish the back-reaction force. They then quote the Penfield and Haus analysis for the mechanical momentum excess possessed by the circulating charges. But, most importantly, they do *not* invoke the incorrect argument of Shockley and James involving the mechanical field momentum. Although there are a few comments in the Coleman and Van Vleck paper, about canonical and mechanical momentum, that the author does not quite agree with, the bulk of the argumentation deals with the two concepts precisely and correctly, and is a joy to read.

4.3 Lagrangian mechanics

In this section, we analyse the various dipoles considered in the previous sections from the viewpoint of Lagrangian mechanics.

4.3.1 The electric dipole

It is actually quite a simple task to obtain the interaction Lagrangian for the electric dipole of Section 4.2.1, from first principles. To do so, one need only note that the *rigid body constraints* that we employed in the definition of that model allow us [96] to simply *add* the interaction Lagrangians of the two individual charges together. Starting with the interaction Lagrangian for each electric charge,

$$L_{\text{int}} = q\Big\{\varphi(\boldsymbol{z}) - \boldsymbol{v}(\boldsymbol{z})\cdot\boldsymbol{A}(\boldsymbol{z})\Big\}\,, \tag{4.61}$$

and using the position and velocity expressions for each charge listed in Section 4.2.1, one quite quickly finds that the sum of the two individual Lagrangians yields

$$L_{\text{int}} = (\boldsymbol{d}\cdot\nabla)(\varphi - \boldsymbol{v}\cdot\boldsymbol{A}) - \dot{\boldsymbol{d}}\cdot\boldsymbol{A}. \tag{4.62}$$

To convert this into a more recognisable form, we use the property, noted in Chapter 2, that one may add a *total time derivative* to the Lagrangian, without affecting its physical content. If we add the total derivative

$$d_t(\boldsymbol{d}\cdot\boldsymbol{A})$$

to (4.62), perform the derivative using the product rule, and employ the convective derivative, we immediately find

$$L_{\text{int}} = \boldsymbol{d}\cdot(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{4.63}$$

If we now differentiate (4.63) with respect to $\boldsymbol{v}$, to obtain the corresponding contribution to the canonical momentum $\boldsymbol{b}$, we find

$$\boldsymbol{b}_{\text{int}} = -\boldsymbol{d} \times \boldsymbol{B}.$$

In other words (adding this result to that of an electric charge), we find

$$\boldsymbol{b} = \boldsymbol{p} + q\boldsymbol{A} - \boldsymbol{d} \times \boldsymbol{B}. \tag{4.64}$$

This indicates that perhaps the principle of "minimal coupling", reviewed in Chapter 2 for an electric charge, might be *extended* when an electric dipole moment is present, to the expression (4.64).

If one computes the Euler–Lagrange equations of motion for the interaction Lagrangian (4.63), one simply finds the same results as found in Section 4.2.1 via Newtonian mechanics, except that, yet again, the nonrelativistic analysis does not yield the redshift force.

If we now compute the *Hamiltonian* for the electric dipole, from the Lagrangian (4.64), one finds

$$H = \frac{(\boldsymbol{b} - q\boldsymbol{A} + \boldsymbol{d} \times \boldsymbol{B})^2}{2m} - \boldsymbol{d} \cdot \boldsymbol{E}. \tag{4.65}$$

Recalling that H is just the zero-component b^0 of the canonical momentum four-vector b^α, we thus obtain the manifestly covariant expression of *extended minimal coupling*:

$$b = p + qA + d \cdot F,$$

or, alternatively,

$$p^2 = (b - qA - d \cdot F)^2 = m^2,$$

of which (4.65) is the nonrelativistic limit.

4.3.2 The magnetic-charge dipole

Since one cannot have both a well-defined four-potential $A(x)$ and magnetic monopoles in the same Universe, without performing serious plastic surgery on the structure of spacetime, one cannot formulate a Lagrangian description of electrodynamics in any simple way if monopoles are employed.

Hence, there is no Lagrangian description of the magnetic-charge dipole.

4.3.3 The electric-current magnetic dipole

If one uses the explicit expressions found in Section 4.2.3, together with the interaction Lagrangian (4.61) for the electric charge, one finds that the electric-current magnetic dipole interaction Lagrangian is simply

$$L_{\text{int}} = \boldsymbol{\mu}\cdot\boldsymbol{B}. \tag{4.66}$$

It will be noted that there is no term in (4.66) dependent on $\boldsymbol{v}$; this follows from the general arguments given in Section 4.2.3. In the context of Lagrangian mechanics, it means that, for the current loop, the canonical three-momentum $\boldsymbol{b}$ *is* simply the same as the mechanical three-momentum $\boldsymbol{p}$. The Euler–Lagrange equations then of course yield

$$\boldsymbol{F} = \nabla(\boldsymbol{\mu}\cdot\boldsymbol{B}),$$

as was found via Newtonian mechanics in Section 4.2.3.

If we compute the Hamiltonian for the current loop, we simply find

$$H = \frac{\boldsymbol{b}^2}{2m} - \boldsymbol{\mu}\cdot\boldsymbol{B}.$$

Now, this result again tells us that there is something strange happening with a current loop: the Hamiltonian (canonical energy) *does* have the term $-\boldsymbol{\mu}\cdot\boldsymbol{B}$ added to it, but the canonical *three-momentum* does not have the relativistic counterpart $\boldsymbol{\mu}\times\boldsymbol{E}$ added to it. This would imply that the canonical four-momentum does not correctly transform under Lorentz transformations.

However, we *know*, from the analysis in Section 4.2.3, that a current loop can either describe a system with constituents that may be “induced” into other configurations—and hence does not properly constitute a system of fixed properties; or, alternatively, if we take the ultra-relativistic limit of the constituents, then we know that we must include the Penfield–Haus effect. Of course, only the latter course of action is of interest to us for the purposes of this thesis. Now, it is not clear how one could modify the Lagrangian description of the electric-current magnetic dipole to incorporate the Penfield–Haus effect. However, since the resulting equations of motion are so similar to the dual of the *electric* dipole results, we might make a *guess* that the interaction Lagrangian

$$L_{\text{int}} \stackrel{?}{=} \boldsymbol{\mu} \cdot (\boldsymbol{B} - \boldsymbol{v} \times \boldsymbol{E}) \tag{4.67}$$

might do the trick. In fact, if one computes the Euler–Lagrange equations due to the interaction Lagrangian (4.67), one finds results in complete agreement with those of the current loop incorporating the Penfield–Haus effect, *including the contact force* $\boldsymbol{\mu} \times \boldsymbol{J}$. Thus, while we have only obtained it by guesswork, it would appear that (4.67) *is* in fact the appropriate Lagrangian for a fixed magnetic dipole. As with the electric dipole case, the canonical momentum now has an extra contribution: all up, we now have

$$\boldsymbol{b} = \boldsymbol{p} + q\boldsymbol{A} - \boldsymbol{d} \times \boldsymbol{B} + \boldsymbol{\mu} \times \boldsymbol{E}. \tag{4.68}$$

The Hamiltonian likewise follows; including all moments, in the nonrelativistic limit, we have

$$H = \frac{(\boldsymbol{b} - q\boldsymbol{A} + \boldsymbol{d} \times \boldsymbol{B} - \boldsymbol{\mu} \times \boldsymbol{E})^2}{2m} - \boldsymbol{d} \cdot \boldsymbol{E} - \boldsymbol{\mu} \cdot \boldsymbol{B}. \tag{4.69}$$

Finally, the principle of extended minimal coupling can be written manifestly covariantly, for all three moments:

$$b = p + qA + d \cdot F + \mu \cdot \widetilde{F},$$

or, alternatively,

$$p^2 = (b - qA - d \cdot F - \mu \cdot \widetilde{F})^2 = m^2.$$

It doesn't come much simpler than that.

4.3.4 Relativistic Lagrangian derivation

We shall now briefly review another method that the author [65] has used to derive the manifestly covariant equations of motion for point particles carrying (fixed) electric charge and electric and magnetic dipole moments. (The full text of this paper is given in Appendix F, but it should be noted that the redshift force does not appear therein, and hence the "uncoupled" equations of motion listed at the end are incorrect.)

In this method of derivation, one essentially recognises from the outset that one wishes the magnitudes of the dipole moments to be *fixed*. One therefore writes down a relativistic Lagrangian in such a way that the dipole moments are treated like *four-vectors of fixed magnitude*. One way to do so, recognising the fundamental practical importance of spin-half particles, is to start from the most general interaction Lagrangian possible between such particles and the electromagnetic field, from the point of view of quantum field theory; and then to massage the functional form of the resulting Lagrangian somewhat. This procedure does not lead to any new results, and is somewhat tangential to the main thrust of this thesis, and so has been relegated to Appendix E. The form of the relativistic Lagrangian found there is

$$L = \frac{1}{2}m(U^2) + q(U\cdot A) + (d\cdot F\cdot U) + (\mu\cdot \widetilde{F}\cdot U).$$

The Euler–Lagrange equations for the four translational degrees of freedom z^α give

$$\begin{aligned} d_\tau(mU) \;&=\; -q d_\tau A - d_\tau(d\cdot F) - d_\tau(\mu\cdot \widetilde{F}) \\ &\quad + q\partial(U\cdot A) + \partial(d\cdot F\cdot U) + \partial(\mu\cdot \widetilde{F}\cdot U). \end{aligned} \tag{4.70}$$

Using the identities (B.27), (B.28) and (B.29), one then immediately obtains the result

$$\dot{p} = qF\cdot U + (d\cdot\partial)F\cdot U + [F\cdot\dot{d}\,] + (\mu\cdot\partial)\widetilde{F}\cdot U + [\widetilde{F}\cdot\dot{\mu}\,] + \mu\times J\times U, \tag{4.71}$$

which can be seen to encompass all of the results of the preceding sections.

As has been noted earlier, the correct procedure when taking the proper-time derivative

$$d_\tau(\Sigma \cdot F)$$

or

$$d_\tau(\Sigma \cdot \widetilde{F})$$

in (4.70) is to use the product rule on the *components* Σ^α, $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$; the former thus yields the *partial* derivative $[\dot{\Sigma}]$ appearing in (4.71), and the latter two yield the convective derivatives

$$(U \cdot \partial) F$$

and

$$(U \cdot \partial) \widetilde{F}.$$

The published paper [65] of Appendix F unfortunately uses instead the covariant derivative $(\dot{\Sigma})$; again, as has been noted in previous sections, this incorrect procedure *removes* the redshift force, which the author now realises *should* be present.

Due to the fact that the author only obtained the redshift force shortly before this thesis was printed, a re-analysis of the uncoupled equations of motion, *à la* that presented in Appendix F, has not yet been performed. However, we intend to perform such an analysis in the near future.

4.4 What does the Dirac equation say?

Throughout this chapter, we have considered the dipole equations of motion purely from the point of view of classical physics. But, as noted in the Abstract of this thesis, the *Dirac equation* also belongs to the field of single-particle electrodynamics: it considers only a single particle, and the electromagnetic field is treated as a classical field. From Ehrenfest's theorem, we know that the operator equations of motion for expectation values should correspond to the classical equations of motion (*see* Section 2.7). But, unlike most applications of this theorem, in *this* case the correct formulation of the classical equations of motion is actually in doubt, whereas the quantum equation is most definitively known. By analysing the Dirac equation, we may therefore gain an independent view of the question of the correct classical limit, as least as far as spin-half particles are concerned.

In Section 4.4.1, we briefly review the physical interpretation of the Foldy–Wouthuysen transformation of the Dirac equation, and explain why it is vital in making contact with the classical limit. Then, in Section 4.4.2, we list the Heisenberg equations of motion arising from the Foldy–Wouthuysen-ed Dirac Hamiltonian. The subtleties involved in the interpretation of these equations are then highlighted in Section 4.4.3, and the expressions compared to those argued for by the author in previous sections.

It should be noted that this section is purely one of review; the author is satisfied that the existing literature covers the physics admirably; but several comments *are* made by the author as to the physical interpretation of the mathematics involved.

For the purposes of this section only, we use units in which $\hbar = 1$.

4.4.1 The Foldy–Wouthuysen transformation

In relativistic quantum mechanics, the wavefunction components for particles and antiparticles are considered together, and indeed may interact with each other. But there always exist *canonical transformations* of the wavefunction (changes of representation) that mix these particle and antiparticle components together, while still leaving the physical quantities represented by the theory unchanged, as long as the operators are complementarily transformed. This means that the components of the wavefunction that appear to represent antiparticles in one representation will actually be a *superposition* of particle and antiparticle components in a different representation.

It would be difficult to recognise a classical limit of the relativistic quantum theory if this arbitrariness in representation were to be permitted to run free. Classical physics does not have any trouble with the concept of antiparticles *per se*: by Feynman's interpretation, antiparticle motion is simply effected by means of the "classical C" transformation

$$\tau \longrightarrow -\tau$$

on the corresponding particle motion (*see* Section A.8.26). But the discreteness of this classical C transformation—and the lack of any sort of "superposition" principle—means that classical physics does *not* admit any "mixing" of particle and antiparticle motion.

The clue to the path out of this dilemma was first found in 1949 by Newton and Wigner [159], as almost a by-product of other, more abstract considerations. The findings of Newton and Wigner eradicated some of the myths surrounding the *position operator* in relativistic wave equations—in particular, that states localised in position cannot be formed solely from positive-energy states; and that if a particle's position is measured below its Compton wavelength, one necessarily generates particle–antiparticle pairs, which renders the position measurement of a single particle impossible. In pursuing some rather simple questions of a group theoretical nature, they not only found what they were looking for, but were greeted with a swag of unexpected bonuses. These were explained and elaborated on by Foldy and Wouthuysen [88], who also obtained the explicit transformation that realised the goals of Newton and Wigner for the physically important case of a spin-half particle. (Case [48] later generalised their method to spin-zero and spin-one particles).

The original aim of Newton and Wigner was to rigorously formulate the properties of *localised states*, for arbitrary-spin relativistic representations of elementary particles. In typical style, they proceeded simply on the basis of *invariance requirements*. They sought a set of states which were localised at a certain point in space, such that any state becomes, after a translation, orthogonal to all of the undisplaced states; such that the superposition of any two such localised states is again a localised state in the set; that the set of states be invariant under rotations about the point of localisation, and under temporal and spatial reflections; and that the states all satisfy certain regularity conditions, amounting to the requirement that all of the operators of the Lorentz group be applicable to them.

From such a simple and reasonable set of requirements, a most bountiful crop was harvested. Firstly, Newton and Wigner found that the set of states they sought *could*, indeed, be found, for arbitrary spin (provided the mass is non-zero); moreover, their requirements in fact specify a *unique* set of states with the desired properties. Furthermore, these states are all *purely positive-energy states* (or, equivalently, purely negative-energy). They further belong to a *continuous eigenvalue spectrum of a particular operator*, which itself has the property of preserving the positive-energy nature of the wavefunction.

Due to these remarkably agreeable properties, Newton and Wigner felt that one would be justified in referring to the operator they had found as *the* position operator—in contradistinction to the operator $\boldsymbol{x}$ in some arbitrary representation of the relativistic wave equation, which only is the "position" operator *in that particular representation*, and hence has no invariant physical meaning—since the representation may be subject to an (in general position-dependent) canonical transformation, that by definition cannot change any physical quantities, but which most definitely changes the expectation values of the fixed operator $\boldsymbol{x}$. The Newton–Wigner position operator had, in fact, been discovered previously in 1935 by Pryce [175], who found the operator a useful tool in the Born–Infeld theory, and again later [176] in a discussion of relativistic definitions of the centre of mass for systems of particles.

A natural question to ask, given the findings of Newton and Wigner, is the following: What does a given relativistic wave equation look like in the representation in which the Newton–Wigner position operator *is*, in fact, simply the three-vector $\boldsymbol{x}$? This is the question effectively asked by Foldy and Wouthuysen in their classic 1950 paper [88], for the physically important case of the Dirac equation. (Their stated aim was actually to find a representation in which the components for positive- and negative-energy states are decoupled, but from the above it is clear that this is effectively the same as seeking the Newton–Wigner representation.) What they in fact found is, even today, simply astounding. Firstly, they found that the canonical transformation from the Dirac–Pauli representation to the Newton–Wigner representation of the *free* Dirac equation is, in fact, obtainable exactly. Secondly, they found that the *Hamiltonian* for the free particle, in the Newton–Wigner representation, agrees completely with that of classical physics,

$$H_{\mathrm{NW}} = \beta(m^2 + \boldsymbol{p}^2)^{1/2} \equiv \beta W_p, \tag{4.72}$$

in contrast to that applicable in the Dirac–Pauli representation,

$$H_{\mathrm{DP}} = \beta m + \boldsymbol{\alpha}\cdot\boldsymbol{p},$$

which—while having the important property of linearity—does not resemble the classical expression at all. (The eigenvalue of the matrix operator β, which takes the values ± 1, is the particle–antiparticle quantum number—effectively, the eigenvalue of C; and since the particle is free, we need not distinguish between the operators $\boldsymbol{b}$ and $\boldsymbol{p}$.)

Thirdly, Foldy and Wouthuysen found that the *velocity operator* (obtained from the position operator by means of its Heisenberg equation of motion) in the Newton–Wigner representation—or, equivalently, the corresponding Newton–Wigner velocity operator in *any* representation—satisfies the *classical* relation for a free particle:

$$\boldsymbol{v}_{\mathrm{NW}} \equiv d_t \boldsymbol{x}_{\mathrm{NW}} = \beta \frac{\boldsymbol{p}}{W_p}. \tag{4.73}$$

That (4.73) is an amazing result is recognised from the fact that, from the very inception of the Dirac equation, it was known that the "velocity" operator in the *Dirac–Pauli* representation *does not* make any classical sense whatsoever: its sole eigenvalues are plus or minus the speed of light; it is not directly related to the mechnical three-momentum $\boldsymbol{p}$; and its equation of motion has non-real "*zitterbewegung*" oscillatory motion (*see*, *e.g.*, [69]). In drastic contradistinction, the Newton–Wigner velocity operator $\boldsymbol{v}_{\mathrm{NW}}$ of (4.73) has the physically understandable continuum of eigenvalues between plus and minus the speed of light; its relationship to the canonical momentum of the free particle is identical to that valid in classical physics; and, when one considers in turn *its* Heisenberg equation of motion, then one finds that, for a free particle, the velocity $\boldsymbol{v}_{\mathrm{NW}}$ is a constant, since $\boldsymbol{p}$ and W are also.

Fourthly, Foldy and Wouthuysen found that the free-particle spin and orbital angular momentum operators in the Newton–Wigner representation—defined to be simply $\boldsymbol{l}_{\mathrm{NW}} \equiv \boldsymbol{x} \times \boldsymbol{p}$ and $\boldsymbol{\sigma}_{\mathrm{NW}} \equiv \boldsymbol{\sigma}$ in this representation—are *constants of the motion separately*; again, it is well-known that, in the Dirac–Pauli representation, these operators are *not* separately constants of the motion, even for a free particle. (The peculiarity of the Dirac–Pauli representation in this respect can, in fact, be traced back to the fact that the "position" operator in that representation exhibits the non-physical "*zitterbewegung*" motion, which thus enters into the motion of the "orbital angular momentum" operator $\boldsymbol{x}_{\mathrm{DP}} \times \boldsymbol{p}$ in this representation.)

As a fifth and final accomplishment, Foldy and Wouthuysen attacked the problem of finding the canonical transformation from the Dirac–Pauli representation to the Newton–Wigner representation, in the case of the *electromagnetically-coupled* Dirac equation. Unfortunately, this cannot be done in closed form. Nevertheless, Foldy and Wouthuysen showed how one can obtain successive approximations to the required transformation, as a power series in $1/m$ (where m is the mass of the particle), for an arbitrary initial

Hamiltonian H_{DP} in the Dirac–Pauli representation.

(An unstated assumption, crucial to the validity of the procedure, is that the "odd" part of the Hamiltonian is in fact of no higher order in m than m^0. This is usually the case, but the assumption has the latent ability to trip one up. For example, if one tries to Foldy–Wouthuysen-transform a Hamiltonian in which the mass term βm has been multiplied by $e^{i\theta\gamma_5}$ (say, by a canonical transformation of the representation), then one can be led to quite erroneous conclusions if one assumes that the terms omitted in the subsequent Foldy–Wouthuysen process are of high order in m; in fact, the omitted terms are of exactly the same order as the terms that are retained; the Foldy–Wouthuysen transformation is, if applied in this way, completely useless. In such cases, the correct procedure is to first perform a simple canonical transformation to remove the order m^{+1} terms from the "odd" parts of the Hamiltonian; the resulting representation may then be fruitfully subjected to the Foldy–Wouthuysen transformation.)

It may be wondered, after hearing of all of the wonderful properties of the Newton–Wigner representation, why one should bother with any other representation at all. In particular, why do we usually only concentrate on the Dirac–Pauli representation of the Dirac equation? (Or representations "trivially" related to it; we shall define this term with more precision shortly.) The answer is subtle, but beautiful. *The massive leptons in Nature are excellently described by a minimal coupling of their Dirac fields to the electromagnetic field, in the Dirac–Pauli representation only.* It is not often stressed that *minimal coupling*—the use of the prescription

$$b \longrightarrow b - qA$$

in the corresponding non-interacting formalism—is *not* a universal, representation-independent transformation. The reason is that, in general, a canonical transformation used to effect a change in representation may be *momentum-dependent*; indeed, the Foldy–Wouthuysen transformation itself is an important example. Clearly, the processes of using minimal coupling, and then performing a momentum-dependent transformation, on the one hand; and that of performing the momentum-dependent transformation first, and *then* using minimal coupling, on the other; will lead to completely *different* relativistic wave equations, in general. *A priori*, *one cannot know which representation one should use the minimal coupling prescription on.*

(Clearly, "trivial" changes of representation, in the sense used above, are therefore those in which the canonical transformation does not involve the canonical momentum operator.)

Dirac was therefore not just brilliant, but also somewhat lucky: he invented a representation, in which to use minimal coupling, that just happened to be *the right representation for the electron* (and, as we now know, the muon and tauon also). But what distinguishes the Dirac representation of spin-half particles from all others? *It gives a free-particle Hamiltonian that is linear in the components of the canonical momentum operator.* This property is, of course, precisely what Dirac was striving for in the first place; even though his reasoning for its *necessity* was, as we now know, flawed, his intuition nevertheless led him along the right path.

We therefore come to recognise that, in reality, there are *two* representations of the Dirac equation that are singled out above all others,—each having qualities unique to itself,—that have a truly direct correspondence with Nature: The Dirac–Pauli representation is unique due to its linearity; it is the representation in which the massive leptons are minimally coupled. The Newton–Wigner representation is unique due to its decoupling of positive- and negative-energy states; it is the representation in which the operators of the free theory correspond to their classical counterparts.

We may go even further, conceptually speaking, in our description of the massive leptons: they are, in effect, two types of particle in the one being: on the one hand, they are four-component animals, in the Dirac–Pauli representation, in which all four components are inextricably coupled, but in which they are *pure, pointlike, structureless electric charges*; on the other hand, they are classical objects, in the Newton–Wigner representation, in which their operators act quite in accord with classical mechanics, but in which their electromagnetic moments are more complicated: they still have electric charge; but through the Foldy–Wouthuysen transformation they acquire a *magnetic moment*, and (less well-known) an *electric charge radius* (manifested in the "Darwin term" in the Hamiltonian; *see* [89, 90, 91, 92]).

(In fact, if one wishes to be rigorous, the Foldy–Wouthuysen transformation of a pure point charge in the Dirac–Pauli representation induces more than just a magnetic moment and an electric charge radius; these are just the lowest-order effects. The *Sachs form factors* [181, 182] are defined precisely so that one may relate the properties of the "Newton–Wigner face" of a fermion—real classical particles measured in real experiments—to the "Dirac–Pauli face" of the fermion—which, in the case of the massive leptons, is a structureless point charge. However, it should be noted that the Sachs form factors effectively only take into account the *lowest order effects* of the Foldy–Wouthuysen transformation, namely, those listed above; despite their usefulness, the physical validity of these form factors—in terms of the above interpretation—cannot be assumed away from the region around $q^2 = 0$. *See* [92] for a thorough discussion of the issues involved.)

4.4.2 The Heisenberg equations of motion

Let us start with the minimally-coupled Hamiltonian in the Dirac–Pauli representation for a pure electric charge q:

$$H_{\text{DP}} = \beta m + \boldsymbol{\alpha}\cdot(\boldsymbol{b} - q\boldsymbol{A}). \tag{4.74}$$

Since we know that the Foldy–Wouthuysen transformation will yield a "pure Dirac" magnetic moment,

$$\boldsymbol{\mu}_{\text{Dirac}} = \frac{q}{2m}\boldsymbol{\sigma}, \tag{4.75}$$

let us therefore add a Pauli term

$$H_{\text{Pauli}} = -\frac{(g-2)q}{8m}\beta\sigma^{\mu\nu}F_{\mu\nu}$$

to the Hamiltonian (4.74) so that the overall magnetic moment of the particle is given by

$$\boldsymbol{\mu}_{\text{total}} = \frac{gq}{4m}\boldsymbol{\sigma} \tag{4.76}$$

with g arbitrary—the pure Dirac result (4.75) applying for $g = 2$. We now apply the Foldy–Wouthuysen transformation to obtain the Hamiltonian in the Newton–Wigner representation; it is straightforward to show (*see*, *e.g.*, [88, 45, 108, 26]) that the result is

$$\begin{aligned} H_{\text{NW}} &= \beta m + \beta\frac{(\boldsymbol{b}-q\boldsymbol{A})^2}{2m} + q\varphi - \frac{gq}{4m}\beta\boldsymbol{\sigma}\cdot\left\{\!\!\left\{\boldsymbol{B} - \frac{g-1}{g}\frac{\beta\boldsymbol{b}}{m}\times\boldsymbol{E}\right\}\!\!\right\} \\ &\quad + \frac{1}{2m}\frac{g-1}{g}\frac{q}{2m}\rho_{\text{ext}} + \mathrm{O}(1/m^2), \end{aligned} \tag{4.77}$$

where we are using the symmetrisor notation of Section A.6.3 to extract physically meaningful operators.

The last term of (4.77) is the Darwin term, which we shall not be interested in here.

The first three terms of (4.77) seem to represent the mechanical energy of the particle, to order $1/m$, plus the electric charge interaction; we shall have more to say on this shortly.

The fourth (symmetrised) term of (4.77), on the other hand, seems to represent a magnetic dipole moment interaction of the particle with the external field, of moment given by (4.76), since the operator $\beta\boldsymbol{b}/m$ is, to lowest order in $1/m$, just the velocity operator. However, there are obvious complications; we shall discuss these in the next section.

Let us, first, simply see what Heisenberg equations of motion are obtained from the Newton–Wigner representation Hamiltonian (4.77). It is straightforward to show [88, 108, 26, 11] that the Heisenberg equation of motion for the position operator $\boldsymbol{x}$ yields

$$\begin{aligned} m d_t \boldsymbol{x} &= \boldsymbol{b} - q\boldsymbol{A} - \frac{g-1}{g}\boldsymbol{\mu}\times\boldsymbol{E} + \mathrm{O}(1/m) \\ &\equiv \boldsymbol{p}_{\mathrm{naive}}; \end{aligned} \tag{4.78}$$

and, in turn, for this operator $\boldsymbol{p}_{\mathrm{naive}}$,

$$\begin{aligned} \boldsymbol{F}_{\mathrm{naive}} &\equiv d_t \boldsymbol{p}_{\mathrm{naive}} \\ &= q\boldsymbol{E} + \frac{g-1}{g}\Big\{(\boldsymbol{\mu}\cdot\nabla)\boldsymbol{B} - \dot{\boldsymbol{\mu}}\times\boldsymbol{E} + \boldsymbol{\mu}\times\boldsymbol{J}_{\mathrm{ext}}\Big\} \\ &\quad + \frac{1}{g}\nabla(\boldsymbol{\mu}\cdot\boldsymbol{B}) + \mathrm{O}(1/m). \end{aligned} \tag{4.79}$$

These results, (4.78) and (4.79), have been collected together and cleaned up somewhat by the author, based on the expressions found by previous workers [88, 108, 26, 11]; but the author has not changed their mathematical content in any respect.

4.4.3 Intepretation of the equations of motion

We now turn to the question of *interpreting* the equations of motion (4.78) and (4.79), and indeed the Hamiltonian (4.77). This question has been considered, over the years, by a number of workers; the author has, for example, found the discussions of Barone [26] and Anandan [11, 12, 14] most helpful; the following conclusions are in large part due to those authors.

The most puzzling aspect of the results (4.77), (4.78) and (4.79), even at first sight, is the recurring presence of the factor

$$\frac{g-1}{g}.$$

Let us first obtain a clear understanding of what this factor represents. In the case of a "pure Dirac" moment, *i.e.*, an *electric charge only* in the Dirac–Pauli representation, we have $g = 2$, and hence

$$\left.\frac{g-1}{g}\right|_{\text{Dirac only}} = \frac{1}{2}.$$

On the other hand, for a *Pauli moment only*, and no electric charge, we effectively have $|g| \to \infty$, and hence

$$\left.\frac{g-1}{g}\right|_{\text{Pauli only}} = 1.$$

Thus, the factor $(g-1)/g$ in front of a term tells us that, for the *anomalous* part of the magnetic moment, the term is fully manifested; but for the *Dirac* part of the magnetic moment, it is reduced by a factor of one-half.

Let us first consider the factor $(g-1)/g$ in the Hamiltonian (4.77): it is the best-known appearance of this factor. It represents, of course, the *Thomas precession* reduction of the spin-orbit coupling of the electron, that allows the $g = 2$ electron to still have the correct energy levels in the hydrogen atom [213, 214]. As such, it is highly desirable, and it was an early success of the Dirac equation that it should yield this relativistic effect automatically.

Naturally, this factor of $(g-1)/g$ propagates through from the Hamiltonian to the Heisenberg equations of motion. Of fundamental concern is its presence in the relation between the canonical and mechanical momentum operators, equation (4.78). (The keeping of terms up to order m^0 essentially places us in the rest frame of the particle; *i.e.*, terms of order $\boldsymbol{v}$ are neglected.) One can already begin to sense the incorrectness of the identification "$\boldsymbol{p}$" for $md_t\boldsymbol{x}$, by realising that a purely *kinematical* effect—the Thomas precession—is purporting to modify the *dynamical* definition of extended minimal coupling [11].

But the clincher comes when one examines in turn the equation of motion for this "$\boldsymbol{p}$" operator, namely, equation (4.79): we find that a proportion $(g-1)/g$ of this equation is given by the accepted equation of motion for a magnetic moment, while the remaining $1/g$ of it is the rejected equation of motion! This nonsensical result is of course a result of the destruction, implied by (4.78), of the extended minimal coupling between $\boldsymbol{p}$ and $\boldsymbol{b}$.

It has been pointed out by Wignall [236] that, *were* this to be true, it would mean that the "pure Dirac" and "anomalous" parts of the magnetic moment would be *classically distinguishable*, in contradiction to all known experiments performed to date, and with perplexing side-effects if the particle in question were to be part of a larger system. For example, if the spin-half particle in question were to be chosen as *a quark inside a nucleon*, such a distinguishability would presumably allow us to measure the relative weights of the *quark* Dirac and anomalous moments simply by measuring the classical motion of the nucleon—a highly dubious possibility. (Incidentally, it was this suggestion by Wignall, linking *nucleon constituent quark* questions to *classical magnetic dipole force* questions, that is responsible for the author's transition from his Fourth Year work on the former, to this Ph.D. thesis on the latter.)

Barone [26] discussed the problem of the factor $(g-1)/g$ most clearly in 1973, highlighting its relationship with the earlier extensive arguments based on "hidden momentum". The stated purpose of his paper was to suggest that the correct definition of the mechanical momentum operator *should*, in fact, be the extended minimal coupling result,

$$\begin{aligned} \boldsymbol{p} &= \boldsymbol{b} - q\boldsymbol{A} - \boldsymbol{\mu}\times\boldsymbol{E}, \\ W &= H - q\varphi + \boldsymbol{\mu}\cdot\boldsymbol{B}, \end{aligned} \tag{4.80}$$

which he noted was a correctly Lorentz-invariant four-vector definition; one can now of course write the Hamiltonian manifestly covariantly, in the form

$$p^2 \equiv (b - qA - \mu\cdot\widetilde{F})^2 \equiv m^2,$$

for which (4.77) (excluding the kinematical Thomas precession term) is the first approximation for $b^0 \equiv H$. (It should by now be apparent why the first three terms of (4.77) do not actually represent the "mechanical energy" of the particle fully: they only take on this rôle for a pure electric charge in the *Newton–Wigner* representation—*i.e.*, the case $g = 0$, where the added Pauli moment actually *cancels* with the Dirac moment.) If one now obtains the Heisenberg equation of motion for the operator $\boldsymbol{p}$ of (4.80), for the Hamiltonian (4.77), one in fact finds that it now agrees with the accepted force law of previous sections; the splitting off of the $1/g$ term disappears. (The redshift force is not present; but we shall show shortly that it vanishes to the order of expansion considered here.)

Anandan [11, 12, 14] also came to the same conclusions as Barone, on the basis of group theoretical considerations, and provided a physical explanation for the discrepancy between (4.78) and (4.80): essentially, because of the Thomas precession of the rest frame of the particle, one must perform a *further* transformation of H_{NW} to take one into the Fermi–Walker-transported coördinate system (which the author has termed the "pre-relativistic" coördinate system in this thesis); in this frame, the relations (4.80) hold rigorously. The motivation stated by Anandan [11] for this clarification was the statement by Goldhaber [94] that the coupling of a quantised spin to the Maxwell field is isomorphic to the interaction of an isospin with the Yang–Mills field; this is reflected in the coupling (4.80), and of course is destroyed in the naïve result (4.78).

It may be thought that the transformation of Anandan puts in jeopardy the original *successful* application of the Thomas precession effect, namely, the calculation of the spin-orbit coupling for the energy levels in the hydrogen atom. However, this is not so: for the hydrogen atom application, the electron is "moving", but the frame we are interested in (the rest frame of the centre of mass of the atom) *is not*; thus, for the purpose of computing energy eigenvalues, it *is* appropriate to use the untransformed Hamiltonian (4.77). Conversely, one may view Thomas's *original* argument for his precession—the difference between viewing the hydrogen atom in the atom's frame and the electron's co-accelerated frame—as showing why the Anandan transformation from the former to the latter is *necessary* for one to obtain relativistically correct Heisenberg equations of motion for the position operator.

If we accept the above line of argumentation by Barone and Anandan—which the author does,—then it is clear that, in the co-accelerated frame, the Dirac equation agrees with the results found in previous sections of this chapter (except for the redshift force), and hence with all previous workers who have suggested the nonrelativistic limit of these results previously. Of course, the controversial "Anandan force" [10, 11, 49], depending only on the *anomalous* moment of the neutron, is supported without qualification by the Dirac equation.

We must now discuss the redshift force found by the author in the closing days of his candidature. Now, since, to lowest order, the acceleration $\dot{\boldsymbol{v}}$ of the particle is given by $q\boldsymbol{E}/m$, and since the expression for the redshift force *itself* contains another explicit factor of $\boldsymbol{E}$, we see that this effect is *quadratic in* $\boldsymbol{E}$; but, more importantly, *it also involves an extra factor of* $1/m$. Thus, the Foldy–Wouthuysen transformation—which is, most rigorously, an expansion in the mathematical parameter $1/m$—would need to be taken to *another* order than that listed above in (4.77), in order for us to obtain the redshift force.

The author has not of course performed this task; the results would no doubt be interesting; we leave it as an exercise for the reader.

Chapter 5

The Retarded Fields

I have preferred to seek an explanation of the facts by supposing them to be produced by actions which go on in the surrounding medium as well as in the excited bodies, and endeavouring to explain the action between distant bodies without assuming the existence of forces capable of acting directly at sensible distances.

—— J. C. Maxwell [149]

5.1 Introduction

The *electromagnetic field* has gained somewhat in status since Maxwell's day. Today, even high school students know that an electric charge is surrounded by an *electric field*; if something *shakes* this charge (say, the voltages in an antenna), then its emits electromagnetic radiation. In order to learn *exactly* what fields are generated by an electric charge, when it is in arbitrary motion, one generally has to do a undergraduate degree in Physics or Electrical Engineering; but, nevertheless, the explicit expressions are there for us, listed in any standard textbook on electromagnetism.

The average high school student would also know that the space around a *magnet* is filled with a *magnetic field*; a piece of paper, some iron filings, and a fridge magnet are all the experimental apparatus necessary to drive this point home quite beautifully. If pressed, such a student would probably also hazard a guess that, if one were to *shake the magnet around vigorously*, some sort of electromagnetic radiation should, by rights, be "shook off"—just as happens with an electric charge. However, an analogue of the radio antenna example is, in this case, not so easy to think of. And what about *exact* expressions for the electomagnetic fields generated by magnets, in *arbitrary* motion: are these in all the standard textbooks? The student would be sadly disappointed if they assumed so.

There is, fundamentally, no good reason why the fields generated by magnetic or electric dipoles, in arbitrary motion, should be treated any differently to those generated by electric charges—other than that of pure lack of interest. Particles carrying dipole moments *do*, naturally, require a somewhat more careful treatment than those simply carrying electric charge—after all, they contain a certain amount of "structure"; but, on the other hand, the particles of Nature, to which we usually apply the equations of classical electrodynamics, generally come with magnetic dipole moments already installed: a fundamental particle *without* a magnetic moment is a rarity.

To repair this deficiency in most textbooks' treatments of the retarded electromagnetic fields, we will, in this chapter, derive explicit expressions for the fields generated by a point particle carrying electric and magnetic dipole moments, as well as electric charge—and, moreover, will obtain them in a very simple form. The mood of the author, in this chapter, is to essentially provide a derivation of the retarded dipole fields that could, with very little work, be grafted on as an extra section in Jackson's textbook [113]—the concepts, methods and notation used essentially mirroring those used in that text.

In Section 5.2, we briefly review the history of the search for the retarded fields for particles with dipole moments. In Section 5.3, we review the derivation of the standard Liénard–Wiechert fields for an electrically charged point particle, both to ground our notation, and to establish the general method of attack on such problems. We then, in Section 5.4, turn these techniques to particles carrying dipole moments, and obtain new, simplified expressions for the results. Finally, in Section 5.5, we review various aspects of the *static* fields from such particles, insofar as are required for the radiation reaction calculations to be undertaken in Chapter 6; as a by-product, we obtain the expression for the extra delta-function field required on the worldline of the particle, in the case of a magnetic moment, in order that the Maxwell equations be correctly satisfied.

5.2 History of the retarded dipole fields

That the fields generated by a particle carrying a dipole moment are richer (and, correspondingly, more mathematically complicated) than those generated by an electric charge can be appreciated before even writing down an equation. Firstly, one knows that a dipole must have *internal degrees of freedom* describing the orientation of the dipole in the rest frame of the particle; these degrees of freedom (and, in particular, their rates of change) will enter into the equations for the generated fields, in addition to those quantities already present in the electric charge case. Secondly, the *static* fields of a dipole fall off like R^{-3} (rather than R^{-2} as for an electric charge), which means that *two* time derivatives of the velocity and/or spin must be present to generate the "radiation" fields. (This can be seen on dimensional grounds: the four-velocity and unit four-spin are themselves dimensionless; there are no other kinematical quantities available apart from the proper time; and the radiation fields must, by definition, fall off like R^{-1} so that the energy it carries may propagate out indefinitely.)

Perhaps due to this premonitory warning of extra complexity, the question of obtaining the general retarded fields for a particle with a dipole moment has not attracted much attention over the decades. Bhabha and Corben [40] appear to have been the first to make a substantial attack on this problem, in 1941, using methods developed two years earlier by Bhabha [39]. (See also [41] and [148] for a discussion of related problems.) Bhabha and Corben obtained exact expressions for the generated fields, which can be found in Appendix A of their paper [40]. Despite (or perhaps because of) their being manifestly-covariant, the *physics* behind the many terms present in the their expressions is not, unfortunately, easy to visualise conceptually.

The field appears to have then lain relatively dormant until the early 1960s. In a series of papers in the period 1963–6, Ellis [79, 80] and Ward [229, 230, 231] proceeded to attack the same problem. (Bialas [43, 44] also considered the case of particles with dipole moments in 1962.) Initially, they appeared to be oblivious not just to each other, but also to their predecessors; but by the end of the series of papers, they had found that their independent analyses essentially agreed with each other (*see* [231]), and with that of Bhabha and Corben (*see* [80]).

The form of the results presented by Ellis was essentially the same as that of Bhabha and Corben (manifestly covariant), but those of Ward were in terms of explicit three-vectors, and lab-time derivatives. The author considers the method of presentation of either author to have both its pros and cons: the manifestly covariant expressions are, as noted, intuitively obscure, but at least their covariance is manifest; the explicit expressions of Ward are somewhat more visualisable, but, unfortunately, the presence of numerous levels of "nested" time-derivatives—not actually evaluated—leaves one again floundering for a simple understanding.

The next attack on the same problem appears to have been that of Kolsrud and Leer [124] in 1967, who concentrated their efforts mainly on the four-potential; the field strengths were obtained in terms of proper-time derivatives, but Kolsrud and Leer only actually computed these derivatives explicitly for the radiation fields, for which they made a number of valuable comments. (They were aware of the previous work of Ward and Ellis, so in a sense there was no need to re-compute the other expressions that they were not interested in.) Again, their results were in accord with those of previous workers.

In 1969, Cohn [53] unfortunately attacked the problem anew, unaware of the previous work in the area. (Somewhat surprisingly, the paper is published in the same journal that carried Ellis's final results just three years earlier.) The reason that Cohn's paper is unfortunate—rather than simply another instance of blissful ignorance—is that, unlike the previous participants in the saga, he got the analysis wrong. This was quickly pointed out by Kolsrud [125] in that same journal.

Fortunately, Cohn did not retreat from the subject; six years later, he and his Ph.D. student Wiebe attacked this problem again [54], making due note of Cohn's earlier mistakes; and, this time, the correct results were obtained. Cohn and Wiebe based their results on the four-potential expression obtained by Kolsrud and Leer [124]; their manifestly-covariant field expressions are, in the opinion of the author, the easiest to come to grips with, from the viewpoint of modern notation and concepts, out of the various treatments listed above (although, mathematically speaking, they are all ultimately equivalent).

As far as the author can ascertain, the field then again lay essentially dormant for another seventeen years, until the author, as blissfully unaware of his predecessors as they were of *their* predecessors, attacked the same problem again, from first principles. The motivation for this was that, following the successful derivation of the dipole equations of motion of Chapter 4, the author wished to use the retarded fields, together with the dipole equations of motion, to obtain the *radiation reaction* equations of motion for particles with dipole moments. Not being, at the time, able to find the retarded fields listed anywhere in the literature (the works cited in this section essentially being a set of measure zero, compared to the total volume of physics literature of the past century), the author proceeded to derive the desired results from scratch.

The results found by the author are presented in the following sections of this chapter. After they were obtained, the author became aware of the Cohn–Wiebe paper. As might be expected, the author had chosen a different set of "convenient quantities" with which to express his results than did Cohn and Wiebe; but, nevertheless, the fundamental quantities used in the computations were basically compatible. (There is considerable freedom in deciding how one is to treat the internal degrees of freedom of the dipole moments; this is partially the reason why some of the earlier results are difficult to interpret, intuitively.) After some work (now presented in Appendix D), the author was able to verify that his manifestly-covariant expressions were, indeed, in all ways equivalent to those of Cohn and Wiebe (and, consequently, to those of their predecessors).

However, the author wished to take the problem one step further: namely, an evaluation of the retarded field expressions in terms of the *explicit, non-covariant* quantities, such as $\boldsymbol{v}$, $\dot{\boldsymbol{v}}$, $\boldsymbol{\sigma}$, $\dot{\boldsymbol{\sigma}}$, *etc.*, that are used to great effect in the Thomas–Bargmann–Michel–Telegdi equation. When the author's attempt towards this end was first begun,—by using the identities, derived by the author, now listed in Section G.4,—the resulting expressions were horrendous. But then, bit by bit, many of the terms began to cancel out with each other; and, after some further pain, the author found a number of additional quantities (specifically, the vectors $\boldsymbol{n}'$ and $\boldsymbol{n}''$, to be introduced later; and, most importantly, the FitzGerald spin vector $\boldsymbol{\sigma}'$) that simplified the results remarkably. The net result is that the expressions found by the author are not only built out of quantities (such as $\dot{\boldsymbol{v}}$) that one can understand intuitively, they furthermore are actually *simpler* than the manifestly-covariant expressions from which they are derived. This was an unexpected bonus.

Finally, while writing the computer algebra program RADREACT to complete the horrendously complicated algebraic computations of the radiation reaction calculations of Chapter 6, the author made a slight detour, and wrote another small program, using the same algebra libraries, to comprehensively check the results of this chapter (*see* Section G.2.2), starting with the manifestly-covariant field expression that had been explicitly verified against those of Cohn and Wiebe.

The program verified the author's explicit results unequivocally. The author therefore confidently asserts that the results are, without doubt, correct.

Now to return to the beginning of the saga, with details filled in.

5.3 The Liénard–Wiechert fields

In this section, we briefly review the computation of the retarded fields generated by a pointlike charged particle in arbitrary motion. Our treatment generally follows that of modern texts; *e.g.*, Jackson [113, Secs. 12.8, 12.11, 14.1]. We emphasise those aspects of the derivation which are to be generalised for the analogous derivation of the retarded dipole fields in Section 5.4.

Note that we do not here consider the fields on the worldline of the generating point particle; this question is considered separately, in Section 5.5.

5.3.1 The field Lagrangian

The classical Lagrangian density for the free electromagnetic field is (*see*, *e.g.*, [113, Sec. 12.8])

$$\mathcal{L}_{\text{free}} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}, \tag{5.1}$$

where, as usual, the field strength tensor $F(x)$ is obtained from the four-potential $A(x)$ by means of the definition (B.1):

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{5.2}$$

The four-potential components $A^\mu(x)$ are considered to be the Lagrangian degrees of freedom for the field, at each spacetime point x.

One's next task is to include the interaction Lagrangian density when electrical sources are present. Pretend, for the moment, that one does not already know the answer, and consider the problem from first principles; this exercise will be of use in guiding us in future sections. We know that, for the purpose of computing a charged *point particle's* equations of motion, the electric charge interaction Lagrangian is

$$L_{\text{int}} = qU^{\alpha}A_{\alpha}. \tag{5.3}$$

We can turn this Lagrangian into a Lagrangian *density* by converting the point particle's electromagnetic current, qU^{α}, into a current density, $J^{\alpha}(x)$, via

$$J^{\alpha}(x) \equiv \int d\tau\, qU^{\alpha}(\tau)\, \delta^{(4)}[x - z(\tau)]\,. \tag{5.4}$$

Then (5.3) is equivalent to the Lagrangian density

$$\mathcal{L}_{\text{int}} = J^{\alpha}A_{\alpha}, \tag{5.5}$$

and so the complete Lagrangian density, as far as the electromagnetic field is concerned, is simply the sum of (5.1) and (5.5):

$$\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} = \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + J^{\alpha}A_{\alpha}. \tag{5.6}$$

5.3.2 The Maxwell equation

A straightforward application of the field Euler–Lagrange equations to the Lagrangian density (5.6) yields the familiar (inhomogeneous) Maxwell equation, equation (B.7) of Section B.2.7:

$$\partial_{\mu}F^{\mu\nu} = J^{\nu}. \tag{5.7}$$

5.3.3 Solution of the Maxwell equation

One's next task is to determine the solution of (5.7) for some given electric current distribution $J^{\nu}(x)$. While it is possible to obtain this information directly from (5.7), it is simpler to employ again the four-potential A^{μ}, *purely* *as a mathematical aid*—it not being, in itself, a physically observable quantity in classical physics. In terms of the A^μ, the Maxwell equation (5.7) is

$$\partial^2 A^\mu - \partial^\mu(\partial_\nu A^\nu) = J^\mu. \tag{5.8}$$

It now simplifies the computations to choose a *Lorentz gauge*, in which

$$\partial_\nu A^\nu_{\text{LG}} = 0. \tag{5.9}$$

This choice, of course, destroys gauge invariance, so we should aim to return to the gauge-invariant (and physically meaningful) field strength tensor $F(x)$ as soon as the possibility arises. With the choice (5.9), equation (5.8) becomes

$$\partial^2 A^\mu_{\text{LG}} = J^\mu. \tag{5.10}$$

The utility of the choice (5.9) lies with the fact that (5.10) now represents four *uncoupled* differential equations in Minkowski space, one for each $\mu = 0, 1, 2, 3$. It therefore suffices to find a Green function $D(x, x')$ satisfying the equation

$$\partial_x^2 D(x, x') = \delta^{(4)}(x - x'). \tag{5.11}$$

Now, this is a merely mathematical problem; the standard textbook solution (*see*, *e.g.*, [113, pp. 609–11] for a derivation) is

$$D_r(x - x') = \frac{1}{2\pi}\vartheta(x^0 - x'^0)\,\delta\Big[(x - x')^2\Big], \tag{5.12}$$

where we have selected the retarded solution, and $\vartheta(t)$ is the Heaviside step function of Section A.5.3. Using this Green function, the solution to (5.10) can be written down directly:

$$A^\mu_{\text{LG}}(x) = A^\mu_{\text{in}}(x) + \int d^4x'\, D_r(x - x') J^\mu(x'), \tag{5.13}$$

where $A^\mu_{\text{in}}(x)$ represents the Lorentz-gauge four-potential of any radiation "incoming", that is not due to the particle in question. Since the equations are linear, we can omit the term $A^\mu_{\text{in}}(x)$ from our explicit considerations, it being understood that the incoming fields may simply be added to the fields generated by the particle in question at the end of the analysis.

5.3.4 The fields for a point particle

The result (5.13) holds for arbitrary electric charge distributions $J^\mu(x')$. We now determine the retarded fields from the electric charge of a *point particle* by substituting the current density expression for a point charge, equation (5.4):

$$A^\mu_{\rm LG}(x) = \frac{q}{2\pi}\int d\tau\, U^\mu(\tau)\int d^4x'\,\vartheta(x^0-x'^0)\,\delta\Big[(x-x')^2\Big]\,\delta^{(4)}[x'-z(\tau)]\,. \quad (5.14)$$

Integrating this expression over d^4x', we find

$$A^\mu_{\rm LG}(x) = \frac{q}{2\pi}\int d\tau\, U^\mu(\tau)\,\vartheta\Big[x^0-z^0(\tau)\Big]\,\delta\Big\{[x-z(\tau)]^2\Big\}\,. \quad (5.15)$$

Clearly, this integral will only contain a contribution from $z(\tau)$ on the backwards light cone of x, *i.e.*, the particular $\tau=\tau_{\rm ret}$ for which $[x-z(\tau_{\rm ret})]^2=0$ and $z^0(\tau_{\rm ret})<x^0$.

We now discard the Lorentz-gauge four-potential $A^\mu_{\rm LG}$ in favour of the physically meaningful field strengths $F_{\alpha\beta}$, using the definition (5.2). Let us first concentrate on the first term on the right-hand side of (5.2), namely, $\partial_\alpha A_\beta$. The partial derivative ∂_α, acting on the Heaviside step function, will give us a Dirac delta function at the four-position of the charge; since we shall be treating the fields on the worldline of the particle separately, in Section 5.5, we can ignore this contribution for the current analysis. We are thus left with

$$\partial_\alpha A_\beta = \frac{q}{2\pi}\int d\tau\, U_\beta(\tau)\,\vartheta\Big[x^0-z^0(\tau)\Big]\,\partial_\alpha\delta\Big\{[x-z(\tau)]^2\Big\}\,. \quad (5.16)$$

To proceed from here, one needs to recall the chain rule of differentiation, namely

$$\partial_u g[f(u,\tau)] \equiv \partial_u f\cdot d_f g[f(u,\tau)]\,. \quad (5.17)$$

Applying this to the case of the partial derivative ∂_α (for which the variable u in (5.17) is the component x^α), and noting that we ultimately wish to integrate over the proper time τ, we write

$$\partial_\alpha g[f(x,\tau)] = \partial_\alpha f \cdot d_f g[f(x,\tau)] \equiv \partial_\alpha f \cdot (d_\tau f)^{-1} \cdot d_\tau g[f(x,\tau)] \,. \tag{5.18}$$

In (5.16) we have $g[f] = \delta[f]$ and $f(x,\tau) = [x - z(\tau)]^2$, and so in this case

$$d_\tau f = -2\, U \cdot (x - z) \tag{5.19}$$

(since $d_\tau z^\alpha(\tau) \equiv U^\alpha(\tau)$), and

$$\partial_\alpha f = 2(x - z)_\alpha,$$

so that

$$\partial_\alpha \delta\Big[(x-z)^2\Big] = -\frac{(x-z)_\alpha}{U \cdot (x-z)}\, d_\tau \delta\Big[(x-z)^2\Big] \,, \tag{5.20}$$

where we are now taking the τ-dependence of z^α and U^α to be understood. Inserting (5.20) into (5.16) then yields

$$\partial_\alpha A_\beta = -\frac{q}{2\pi} \int d\tau \, \frac{(x-z)_\alpha U_\beta}{U \cdot (x-z)}\, \vartheta(x^0 - z^0)\, d_\tau \delta\Big[(x-z)^2\Big] \,. \tag{5.21}$$

We can now compute the field strength tensor (5.2) by antisymmetrising (5.21). Performing the τ integration by parts, the boundary term vanishes on account of the delta function, and we are left with

$$F^q = \frac{q}{2\pi} \int d\tau \, d_\tau \left[\frac{(x-z) \wedge U}{U \cdot (x-z)} \right] \vartheta(x^0 - z^0)\, \delta\Big[(x-z)^2\Big] \,,$$

where $\wedge$ represents the wedge-product operation of Section A.8.11. We further make use of the fact that, for purposes of integration over τ,

$$\delta[f(\tau)] = \sum_{\tau_z} \delta(\tau - \tau_z)\, |d_\tau f(\tau)|_{\tau_z}^{-1} \,, \tag{5.22}$$

where τ_z are the zeroes of $f(\tau)$. Using (5.19), we thus have

$$F^q = \frac{q}{4\pi\, [\, U \cdot (x-z)\,]_{\tau_{\rm ret}}}\, d_\tau \left[\frac{(x-z) \wedge U}{U \cdot (x-z)} \right]_{\tau_{\rm ret}} \,, \tag{5.23}$$

where, again, $\tau_{\rm ret}$ is the retarded proper time.

It is convenient, both here and with an eye to the next section, to define three new quantities ζ^α, φ and $\dot{\chi}$, according to

$$\zeta^\alpha \equiv (x-z)^\alpha, \tag{5.24}$$

$$\varphi \equiv \frac{1}{U\cdot(x-z)} \equiv \frac{1}{(U\cdot\zeta)}, \tag{5.25}$$

$$\dot{\chi} \equiv \dot{U}\cdot(x-z) \equiv (\dot{U}\cdot\zeta). \tag{5.26}$$

(Note that $\dot{\chi}$ is *not* actually the derivative of any previously defined quantity χ; the overdot added to the symbol will be convenient for our later purposes, and is motivated by the fact that $\dot{\chi}$ involves the quantity $\dot{U}$.) Equation (5.23), rewritten in terms of ζ and φ, is

$$4\pi F^q = q\varphi d_\tau\Big\{\varphi\zeta\wedge U\Big\}.$$

Differentiation by the product rule, and a collecting together of terms, yields

$$4\pi F^q = q\varphi\dot{\varphi}\zeta\wedge U + q\varphi^2\zeta\wedge\dot{U},$$

where we have noted that

$$d_\tau\zeta \equiv -d_\tau z \equiv -U,$$

and we have used the identity

$$U\wedge U \equiv 0.$$

Now, differentiating (5.25), we find that

$$\dot{\varphi} = \varphi^2(1-\dot{\chi}), \tag{5.27}$$

so that

$$4\pi F^q = q\varphi^3(1-\dot{\chi})\zeta\wedge U + q\varphi^2\zeta\wedge\dot{U}. \tag{5.28}$$

Equation (5.28) is, in the above notation, the manifestly covariant expression for the retarded fields generated by the point charge. It is, however, useful to further split it into two parts: F_1^q, which has a dependence on ζ of ζ^{-1}; and F_2^q, which has a dependence on ζ of ζ^{-2}; while we are at it, we shall also deal with the ever-inconvenient factor 4π, and the value of the electric charge, q:

$$F^q \equiv \frac{q}{4\pi}\Big\{F_1^q + F_2^q\Big\}.$$

An examination of (5.28), and the definitions (5.24), (5.25) and (5.26), allows us to slot the terms of (5.28) into F_1^q and F_2^q:

$$\begin{aligned} F_2^q &= \varphi^3 \zeta \wedge U, \\ F_1^q &= \varphi^2 \zeta \wedge \dot{U} - \varphi^3 \dot{\chi} \zeta \wedge U. \end{aligned} \tag{5.29}$$

5.3.5 Explicit form for the retarded fields

While being mathematically elegant, it is somewhat difficult to appreciate the true physical content of the expressions (5.29), as they stand. One therefore usually reëxpresses them in terms of *explicit, non-covariant quantities*: the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in the case of $F_{\alpha\beta}$, and the kinematical quantities $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ in the case of U^α and $\dot{U}^\alpha$. The lightlike four-separation $\zeta^\alpha \equiv (x-z)^\alpha$ between the observation point x^α and the retarded lightcone four-position z^α may be parametrised as

$$\begin{aligned} (x-z)^0 &\equiv R, \\ (\boldsymbol{x}-\boldsymbol{z}) &\equiv R\boldsymbol{n}, \end{aligned} \tag{5.30}$$

where $\boldsymbol{n}^2 = 1$. Clearly, $\boldsymbol{n}$ can be interpreted as the unit normal in the direction of the observation point, from the position of the charge as it was at the *retarded* time τ_{ret}; R is the simple three-distance between these temporally-separated events. From the relations listed in Section G.4.7, and the definitions (5.25) and (5.26), one can verify that

$$\dot{\chi} = \gamma^3\varphi^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) - \gamma^2 R(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}), \tag{5.31}$$

$$\varphi = \frac{1}{\gamma R\,[1-\boldsymbol{v}\cdot\boldsymbol{n}]}. \tag{5.32}$$

(These results are also verified explicitly by the computer algebra program KINEMATS, in Section G.4.11.) We can now compute the components of electric and magnetic field, given by $\boldsymbol{E} \equiv F^{i0}$ and $\boldsymbol{B} \equiv -\frac{1}{2}\varepsilon_{ijk}F^{jk}$. Performing the relevant substitutions in (5.28), and manipulating the results somewhat using vector identities, one finds that

$$\boldsymbol{E}^q = \frac{q\,[\boldsymbol{n}-\boldsymbol{v}]}{4\pi\gamma^2[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3R^2} + \frac{q\,\boldsymbol{n}\times([\boldsymbol{n}-\boldsymbol{v}]\times\dot{\boldsymbol{v}})}{4\pi[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3R}, \tag{5.33}$$

where it is understood that the particle quantities $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ are evaluated at the retarded time, τ_{ret}. The result (5.33) is in the form presented, for example, in Jackson's textbook [113, Eq. (14.14)]; it is described there as "inelegant, but perhaps more intuitive" than its manifestly-covariant counterpart. We shall attempt to improve its "elegance rating" somewhat, shortly.

To compute the retarded magnetic field $\boldsymbol{B}^q$, one need not start from scratch, but rather one may make good use of the identities in Section G.4. By writing out the expressions explicitly, one can verify that, if a term of F is proportional to U (wedged with some other four-vector), then, for that term,

$$\boldsymbol{B} = \boldsymbol{v}\times\boldsymbol{E}; \tag{5.34}$$

alternatively, if a term of F is proportional to ζ (again, wedged with some other four-vector), then, for that term,

$$\boldsymbol{B} = \boldsymbol{n}\times\boldsymbol{E}. \tag{5.35}$$

Clearly, all of (5.28) satisifes (5.35), and the first term also satisfies (5.34). Using the former of these properties, one may most simply write

$$\boldsymbol{B}^q = \boldsymbol{n}\times\boldsymbol{E}^q; \tag{5.36}$$

this is Jackson's equation (14.13) [113].

With an eye to the next section, we now define the convenient quantities $\boldsymbol{E}_1'^q$, $\boldsymbol{E}_2'^q$, $\boldsymbol{B}_1'^q$ and $\boldsymbol{B}_2'^q$:

$$\begin{aligned} \boldsymbol{E}^q &\equiv \frac{q}{4\pi}\left\{\frac{\boldsymbol{E}_1'^q}{R} + \frac{\boldsymbol{E}_2'^q}{R^2}\right\}, \\ \boldsymbol{B}^q &\equiv \frac{q}{4\pi}\left\{\frac{\boldsymbol{B}_1'^q}{R} + \frac{\boldsymbol{B}_2'^q}{R^2}\right\}, \end{aligned} \tag{5.37}$$

as well as the (unprimed) quantities $\boldsymbol{E}_1^q$, $\boldsymbol{E}_2^q$, $\boldsymbol{B}_1^q$ and $\boldsymbol{B}_2^q$:

$$\begin{aligned} \boldsymbol{E}_n^q &\equiv \frac{\boldsymbol{E}_n'^q}{R^n}, \\ \boldsymbol{B}_n^q &\equiv \frac{\boldsymbol{B}_n'^q}{R^n}. \end{aligned}$$

It will also be fruitful to define the two new three-vectors

$$\boldsymbol{n}' \equiv \boldsymbol{n} - \boldsymbol{v} \tag{5.38}$$

and

$$\boldsymbol{n}'' \equiv \boldsymbol{n}' - \boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v}), \tag{5.39}$$

as well as the "Doppler factor"

$$\kappa \equiv \frac{1}{1 - (\boldsymbol{v}\cdot\boldsymbol{n})}. \tag{5.40}$$

In terms of these quantities, the retarded electric charge fields (5.33) and (5.36) can be written

$$\boldsymbol{E}_1'^q = \kappa^3 \boldsymbol{n}\times(\boldsymbol{n}'\times\dot{\boldsymbol{v}}), \tag{5.41}$$

$$\boldsymbol{E}_2'^q = \gamma^{-2}\kappa^3 \boldsymbol{n}', \tag{5.42}$$

and

$$\boldsymbol{B}_1'^q = \boldsymbol{n}\times\boldsymbol{E}_1'^q, \tag{5.43}$$

$$\boldsymbol{B}_2'^q = \boldsymbol{n}\times\boldsymbol{E}_2'^q = \boldsymbol{v}\times\boldsymbol{E}_2'^q, \tag{5.44}$$

where in (5.44) we have noted that $\boldsymbol{B}_2^q$ (and hence $\boldsymbol{B}_2'^q$) satisfies both (5.34) and (5.35).

Equations (5.41), (5.42), (5.43) and (5.44) are the final, simplified expressions for the retarded fields generated by a point electric charge, in the author's notational system.

5.4 The retarded dipole fields

In this section, we compute the retarded electromagnetic fields generated by an arbitrarily moving particle possessing *magnetic and electric dipole moments*, in addition to electric charge.

It will be found convenient to assume from the outset that the dipole moments are *fixed* in magnitude, such as is true for the intrinsic moments of spin-half particles. We may then employ the *unit spin four-vector* Σ^α of Section G.4.4 to denote the direction of the dipole moment from the outset, with a constant numerical coëfficients d and μ. However, it should be noted that the *angular momentum* connotation of Σ is *not* used at all, in the considerations of this chapter. In other words, the results are *completely applicable* to classical dipoles in which the (fixed) dipole moment vector and the spin angular momentum vector have no definite relationship; one should, in these cases, simply replace $\boldsymbol{\sigma}$, in the results of this chapter, by $\boldsymbol{d}$ or $\boldsymbol{\mu}$ as appropriate.

5.4.1 The field Lagrangian

Our first step is to return to the classical free-field Lagrangian density of the electromagnetic field, equation (5.1):

$$\mathcal{L}_{\text{free}} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta}.$$

To this Lagrangian density we must add terms corresponding to the interaction between the electromagnetic field and the electric and magnetic dipole moments of a particle. There are a number of ways of doing this; here, we follow the same method as was used in the Section 5.3.1: we start with the interaction Lagrangian for a *point particle* carrying such moments. From equation (E.7) of Section E.4, we have

$$L_{\text{int}} = qU^\mu A_\mu + \mu^\alpha \widetilde{F}_{\alpha\beta} U^\beta + d^\alpha F_{\alpha\beta} U^\beta,$$

We now need to generalise the latter two terms to the case of arbitrary dipole *densities*, analogous to the generalisation, in Section 5.3.1, of the first term from charged current to charged current density. To this end, we define two antisymmetric density tensors $\tilde{d}^{\alpha\beta}(x)$ and $\tilde{\mu}^{\alpha\beta}(x)$ for the point particle, thus:

$$\tilde{d}(x) \equiv \int d\tau\, U(\tau) \wedge d(\tau)\, \delta^{(4)}[\, x - z(\tau)\,], \tag{5.45}$$

$$\tilde{\mu}(x) \equiv \int d\tau\, U(\tau) \wedge \mu(\tau)\, \delta^{(4)}[\, x - z(\tau)\,]. \tag{5.46}$$

The combined field–particle Lagrangian density may then be written

$$\mathcal{L} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + J^\alpha A_\alpha - \frac{1}{2} \tilde{\mu}^{\alpha\beta} \widetilde{F}_{\alpha\beta} - \frac{1}{2} \tilde{d}^{\alpha\beta} F_{\alpha\beta}. \tag{5.47}$$

5.4.2 The dipole current density tensor

It is possible to simplify the last two terms of (5.47) a little further. Expanding out the dual field strength tensor in terms of its definition (B.3), $\widetilde{F}_{\alpha\beta} \equiv -\frac{1}{2}\varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu}$, we have

$$-\frac{1}{2} \tilde{\mu}^{\alpha\beta} \widetilde{F}_{\alpha\beta} - \frac{1}{2} \tilde{d}^{\alpha\beta} F_{\alpha\beta} = \frac{1}{4} \tilde{\mu}^{\alpha\beta} \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu} - \frac{1}{2} \tilde{d}^{\alpha\beta} F_{\alpha\beta}.$$

Relabelling indices, this last expression becomes

$$\frac{1}{4} \tilde{\mu}_{\mu\nu} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} - \frac{1}{2} \tilde{d}^{\alpha\beta} F_{\alpha\beta}.$$

We now define the *dipole current density tensor*, $\tilde{J}^{\alpha\beta}(x)$, as

$$\tilde{J}^{\alpha\beta}(x) \equiv \tilde{d}^{\alpha\beta}(x) - \frac{1}{2} \varepsilon^{\alpha\beta\mu\nu} \tilde{\mu}_{\mu\nu}(x). \tag{5.48}$$

In terms of $\tilde{J}(x)$, the complete field Lagrangian density (5.47) then takes its simplest form:

$$\mathcal{L} = \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} + J^{\alpha} A_{\alpha} - \frac{1}{2} \tilde{J}^{\alpha\beta} F_{\alpha\beta}. \tag{5.49}$$

5.4.3 The generalised Maxwell equation

We now compute the Euler–Lagrange equations for the electromagnetic potentials using the Lagrangian density (5.49). The canonical field $\Pi_{\mu\nu}(x)$ conjugate to A^{α} is given by

$$\Pi_{\mu\nu} \equiv \partial_{\partial^{\mu} A^{\nu}} \mathcal{L} = F_{\mu\nu} - \tilde{J}_{\mu\nu}. \tag{5.50}$$

(Explicitly computed, only the combination $\tilde{J}_{[\mu\nu]}$ contributes to $\Pi_{\mu\nu}$; it is basically for this reason that we were free to arbitrarily define $\tilde{J}$ to be antisymmetric in (5.48).) Furthermore, we have

$$\partial_{A^{\nu}} \mathcal{L} = J_{\nu}.$$

The field Euler–Lagrange equations for the fields $A^{\mu}(x)$ thus yield

$$\partial^{\mu} \partial_{\partial^{\mu} A^{\nu}} \mathcal{L} - \partial_{A^{\nu}} \mathcal{L} = \partial^{\mu} \Big(F_{\mu\nu} - \tilde{J}_{\mu\nu} \Big) - J_{\nu} = 0,$$

or, upon reärranging, and employing component-free notation,

$$\partial \cdot F = J + \partial \cdot \tilde{J}. \tag{5.51}$$

Equation (5.51) is the *generalised inhomogeneous Maxwell equation* for the case of both monopolar (*i.e.*, electric charge) and dipolar sources, for applications in which it is convenient to separate the former from the latter. (Fundamentally, of course, the dipole moments *are* implicitly included in the source term $J^{\alpha}(x)$ of the regular inhomogeneous Maxwell equation (5.7); the source vector J^{α} appearing in (5.51) should properly be referred to as that due to *free charge only*.)

It can be noted that equation (5.51) includes, as a special case, the Maxwell–Lorentz theory of macroscopic fields, for which $J^{\alpha\beta}$ represents the "average magnetisation" and "average polarisation" of the particles comprising the "medium". However, (5.51) is, as a differential equation, in fact exact *microscopically* (within the domain of the classical limit, of course), and does *not* require any "averaging" for its validity.

(To make this connection complete, one needs to know that a careful re-analysis shows that, while we have in the above assumed the dipole moments to be *fixed* in magnitude, the equation (5.51) is, in fact, true for *any* dipole source densities—whether fixed, or "induced" by external fields. However, this property lies beyond the requirements of this thesis; and we will, shortly, introduce assumptions into our analysis that will again restrict the validity of the results to particles with fixed moments only.)

5.4.4 Solution of the generalised Maxwell equation

We now seek a general solution to (5.51) for arbitrary free charge and dipole densities, $J(x)$ and $\tilde{J}(x)$. Again, we seek to phrase the problem in such a way that a minimal amount of new work must be performed. Employing the four-potential A^μ, as we did for the analogous problem in Section 5.3.3, (5.51) becomes

$$\partial^2 A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu + \partial_\mu \tilde{J}^{\mu\nu}.$$

In Section 5.3.3, we employed a Lorentz gauge, characterised by the condition $\partial_\mu A^\mu_{\mathrm{LG}} = 0$, to simplify the left hand side; here, that choice yields

$$\partial^2 A^\nu_{\mathrm{LG}} = J^\nu + \partial_\mu \tilde{J}^{\mu\nu},$$

or, on dropping the free electric charge source term J^ν (which we have already treated in Section 5.3),

$$\partial^2 A^\nu_{\mathrm{LG}} = \partial_\mu \tilde{J}^{\mu\nu}. \tag{5.52}$$

There are at least two ways in which we could proceed from equation (5.52). The most obvious path would be to use the Green function solution (5.12) of (5.11) for each of the four values of ν. This then requires that we integrate by parts, shifting the derivative operator $\partial_{\mu'}$ to $D_r(x - x')$, and then use its antisymmetry to convert this into ∂_μ, and thence into proper-time derivatives.

Instead, we follow a slightly different procedure, which shows most explicitly how the current derivation parallels that of equation (5.10): we introduce an additional mathematical aid which, like the four-potential A^μ, is unobservable physically: the *tensor potential* $\tilde{A}^{\alpha\beta}(x)$. It is defined by two properties. Firstly, we deem that it satisfies the equation

$$\partial^2 \tilde{A}^{\alpha\beta} = \tilde{J}^{\alpha\beta}. \tag{5.53}$$

Secondly, we deem that the Lorentz-gauge four-potential $A^\nu_{\rm LG}$ be derivable from the tensor potential $\tilde{A}^{\alpha\beta}$ by means the relation

$$A^\nu_{\rm LG} \equiv \partial_\mu \tilde{A}^{\mu\nu}. \tag{5.54}$$

It is clear that, if a tensor function $\tilde{A}^{\alpha\beta}(x)$ can be found that satisfies the equation (5.53), then the definition (5.54) ensures that the derived Lorentz-gauge four-potential $A^\alpha_{\rm LG}(x)$ will satisfy (5.52); and hence the field strength tensor $F_{\alpha\beta}(x)$ derived in turn from $A^\alpha_{\rm LG}$ will satisfy the inhomogeneous dipolar part of the generalised Maxwell equation (5.51). Thus, our work has been reduced to finding a solution to (5.53) for a given dipole current density $\tilde{J}^{\alpha\beta}(x)$, which is the tensor analogue of the derivation proceeding from (5.10). A general solution can therefore be written down immediately:

$$\tilde{A}^{\mu\nu}(x) = A^{\mu\nu}_{\rm in}(x) + \int d^4x'\, D_r(x - x') \tilde{J}^{\mu\nu}(x'), \tag{5.55}$$

where $D_r(x-x')$ is given by (5.12). The retarded fields generated by a particle carrying both electric charge and dipole moments can thus be computed via the *overall* Lorentz-gauge four-potential $A^\mu_{\rm LG}(x)$, now given by

$$A^\mu_{\rm LG}(x) = A^\mu_{\rm charge}(x) + \partial_\nu \tilde{A}^{\nu\mu}(x),$$

where $A^\mu_{\rm charge}(x)$ is given by equation (5.13) of Section 5.3.3.

5.4.5 The fields for a point particle

We can now determine the retarded fields that are generated by a classical *point particle* carrying electric and magnetic dipole moments. As noted earlier, we are, for practical simplicity, parametrising the dipole moment four-vectors d^α and μ^α in terms of the unit four-spin vector $\varSigma^\alpha$:

$$\begin{aligned} d^\alpha &= d\,\varSigma^\alpha, \\ \mu^\alpha &= \mu\,\varSigma^\alpha, \end{aligned}$$

where the rest-frame dipole moment magnitudes d and μ are fixed. The analogue of (5.4) can then be written down by using equations (5.45), (5.46) and (5.48):

$$\tilde{J}^{\alpha\beta}(x) = \left(d\, g^{\alpha\mu} g^{\beta\nu} - \frac{1}{2}\mu\, \varepsilon^{\alpha\beta\mu\nu} \right) \int d\tau\, \delta^{(4)}[\, x - z(\tau)\,]\, U_{[\mu}(\tau) \varSigma_{\nu]}(\tau). \quad (5.56)$$

Substituting (5.12) and (5.56) into (5.55), and integrating over d^4x', we find

$$\tilde{A}^{\alpha\beta}(x) = \left(\frac{d}{2\pi} g^{\alpha\mu} g^{\beta\nu} - \frac{\mu}{4\pi} \varepsilon^{\alpha\beta\mu\nu} \right) \int d\tau\, U_{[\mu} \varSigma_{\nu]} \vartheta(x^0 - z^0)\, \delta\Big[(x - z)^2\Big], \quad (5.57)$$

which is the analogue of (5.15).

We now discard the tensor potential $\tilde{A}^{\mu\nu}(x)$—and, indeed, the Lorentz-gauge four-potential $A^\mu_{\rm LG}(x)$ itself—in favour of the field strength tensor $F_{\alpha\beta}(x)$. From (5.2) and (5.54) we have

$$F_{\alpha\beta} \equiv \partial_\alpha \partial_\mu A^\mu{}_\beta - \partial_\beta \partial_\mu A^\mu{}_\alpha. \quad (5.58)$$

It is clear that, to carry out the partial derivatives present in (5.58), we shall need to compute derivatives such as

$$\partial_\alpha \partial_\beta g[f(x,\tau)]\,, \tag{5.59}$$

where $g[f]$ is either a Heaviside step function or Dirac delta function. Clearly, the derivatives of the step function will yield terms that are non-vanishing only on the worldline, which we are again ignoring in this section. Now, from (5.17) we have

$$\partial_u \partial_v g[f(u,v,\tau)] = \partial_v \Big\{ \partial_u f \cdot d_f g[f(u,v,\tau)] \Big\}\,.$$

Carrying out the v differentiation by the product rule, we obtain

$$\partial_u \partial_v f \cdot d_f g[f] + \partial_u f \cdot \partial_v \Big(d_f g[f] \Big)\,.$$

To compute this last term, one can consider $d_f g[f]$ as some new function, $h[f]$, upon which we perform the same process as carried out in (5.17). One then finds that

$$\partial_u \partial_v g[f(u,v,\tau)] = \partial_u \partial_v f \cdot d_f g[f] + \partial_u f \cdot \partial_v f \cdot d_f^2 g[f]. \tag{5.60}$$

We now proceed to replace the d_f derivatives in (5.60) by derivatives with respect to τ. We again use the chain rule, namely

$$\begin{aligned} d_f &\equiv d_f\tau \cdot d_\tau, \\ d_f^2 \equiv d_f d_f &\equiv d_f^2\tau \cdot d_\tau + (d_f\tau)^2 \cdot d_\tau^2. \end{aligned}$$

From the above, it is also clear that

$$d_f^2\tau \equiv -d_\tau^2 f \cdot (d_\tau f)^{-3}.$$

Using these identities in (5.59), we finally obtain the desired relation:

$$\begin{aligned} \partial_\alpha \partial_\beta g[f] \;=\; & \partial_\alpha \partial_\beta f \cdot (d_\tau f)^{-1} \cdot d_\tau g[f] \\ & + \partial_\alpha f \cdot \partial_\beta f \cdot (d_\tau f)^{-2} \cdot d_\tau^2 g[f] \\ & - \partial_\alpha f \cdot \partial_\beta f \cdot d_\tau^2 f \cdot (d_\tau f)^{-3} \cdot d_\tau g[f]. \end{aligned} \tag{5.61}$$

Applying (5.61) to the delta function in (5.57), we have $g[f] = \delta[f]$ and $f(x,\tau) = (x-z)^2 \equiv \zeta^2$. The relevant derivatives are then given by

$$\begin{aligned}
\partial_\alpha f &= 2(x-z)_\alpha \equiv 2\zeta_\alpha, \\
\partial_\alpha \partial_\beta f &= 2g_{\alpha\beta}, \\
d_\tau f &= -2\,U\cdot(x-z) \equiv -2\varphi^{-1}, \\
d_\tau^2 f &= 2\Big[1 - \dot{U}\cdot(x-z)\Big] \equiv 2(1-\dot{\chi}),
\end{aligned}$$

which, in (5.61), yield

$$\partial_\alpha \partial_\beta\, \delta(\zeta^2) = -\varphi g_{\alpha\beta} d_\tau \delta(\zeta^2) + \varphi^2 \zeta_\alpha \zeta_\beta d_\tau^2 \delta(\zeta^2) + \varphi^3 \zeta_\alpha \zeta_\beta (1-\dot{\chi}) d_\tau \delta(\zeta^2). \quad (5.62)$$

Inserting this result into (5.57), performing the τ integration by parts (twice, for the last term of (5.62)), and using (5.22) and (5.58), we find

$$F = F^d + F^\mu,$$

where

$$\begin{aligned}
4\pi F^d &= \varphi d_\tau \{2\varphi U \wedge \Sigma\} \\
&\quad - \varphi d_\tau \Big\{\varphi^3(1-\dot{\chi})\zeta \wedge \Big(\varphi^{-1}\Sigma - (\Sigma\cdot\zeta)U\Big)\Big\} \\
&\quad + \varphi d_\tau^2 \Big\{\varphi^2 \zeta \wedge \Big(\varphi^{-1}\Sigma - (\Sigma\cdot\zeta)U\Big)\Big\}, \qquad (5.63)
\end{aligned}$$

and

$$\begin{aligned}
4\pi F^\mu &= \varphi d_\tau \Big\{2\varphi U \times \Sigma\Big\} \\
&\quad - \varphi d_\tau \Big\{\varphi^3(1-\dot{\chi})\zeta \wedge (\zeta \times \Sigma \times U)\Big\} \\
&\quad + \varphi d_\tau^2 \Big\{\varphi^2 \zeta \wedge (\zeta \times \Sigma \times U)\Big\}. \qquad (5.64)
\end{aligned}$$

We concentrate first on the fields generated by an electric dipole moment d. Defining the symbol ψ as

$$\psi \equiv (\Sigma\cdot\zeta), \qquad (5.65)$$

equation (5.63) can be written

$$\begin{aligned}4\pi F^d &= \varphi d_\tau \Big\{ 2\varphi U\wedge\Sigma - \varphi^2(1-\dot\chi)\zeta\wedge\Sigma - \varphi^3(1-\dot\chi)\psi\zeta\wedge U \Big\} \\ &\quad + \varphi d_\tau^2 \Big\{ \varphi\zeta\wedge\Sigma - \varphi^2\psi\zeta\wedge U \Big\}.\end{aligned}$$

Making the further definitions

$$\ddot\chi \equiv d_\tau\dot\chi \equiv [\zeta\cdot\ddot U], \tag{5.66}$$

$$\dot\vartheta \equiv [U\cdot\dot\Sigma], \tag{5.67}$$

$$\ddot\eta \equiv [\zeta\cdot\ddot\Sigma], \tag{5.68}$$

and noting that

$$d_\tau\dot\psi \equiv [\zeta\cdot\ddot\Sigma] - [U\cdot\dot\Sigma] \equiv \ddot\eta - \dot\vartheta,$$

one finds

$$F^d = \frac{d}{4\pi}\Big\{ F_1^d + F_2^d + F_3^d \Big\}, \tag{5.69}$$

where

$$F_3^d = \varphi^3 U\wedge\Sigma - 3\varphi^5\psi\zeta\wedge U \tag{5.70}$$

$$\begin{aligned}F_2^d &= \varphi^2\dot U\wedge\Sigma + \varphi^3[\zeta\wedge\dot\Sigma] + \varphi^3\psi U\wedge\dot U - \varphi^3\dot\chi U\wedge\Sigma \\ &\quad + 6\varphi^5\dot\chi\psi\zeta\wedge U - 3\varphi^4\dot\psi\zeta\wedge U - 3\varphi^4\psi\zeta\wedge\dot U + \varphi^3\dot\vartheta\zeta\wedge U\end{aligned} \tag{5.71}$$

$$\begin{aligned}F_1^d &= \varphi^2[\zeta\wedge\ddot\Sigma] - \varphi^3\psi[\zeta\wedge\ddot U] + \varphi^4\psi\ddot\chi\zeta\wedge U - 2\varphi^3\dot\psi\zeta\wedge\dot U + 3\varphi^4\dot\chi\dot\psi\zeta\wedge U \\ &\quad - \varphi^3\dot\chi[\zeta\wedge\dot\Sigma] - \varphi^3\ddot\eta\zeta\wedge U - 3\varphi^5\dot\chi^2\psi\zeta\wedge U + 3\varphi^4\psi\dot\chi\zeta\wedge\dot U.\end{aligned} \tag{5.72}$$

The order of R of each of term in (5.70), (5.71) and (5.72) may be verified by noting that φ is of order R^{-1}; $\dot\vartheta$, U, Σ, and overdots are of order R^0; and $\ddot\eta$, $\dot\chi$, ψ and ζ are of order R^1. It will be noted that all terms in F_3^d, F_2^d and F_1^d have zero, one and two overdots respectively, in agreement with the dimensional argument presented in Section 5.1. (It is to simplify this bookkeeping that the symbols $\dot\chi$, $\dot\vartheta$ and $\ddot\eta$ were defined in such a way so as to "conserve the number of overdots".)

The results above are for a particle with an electric dipole moment d. For a magnetic dipole moment μ, one could likewise proceed from equation (5.64), just as we proceeded above from equation (5.63). It may be verified that this computation simply gives us the electromagnetic duals of the terms appearing in (5.70), (5.71) and (5.72); we shall not explicitly write out the detailed analysis here.

Since the explicit results (5.70), (5.71) and (5.72) are not widely known, and since a significant amount of further analysis in this thesis is based upon them, it is desirable to verify that they are indeed correct. In Appendix D, we show that the results above are, in fact, identical to those obtained by Cohn and Wiebe [54], and hence to those of all workers who have considered this question (*see* the discussion in Section 5.2).

5.4.6 Explicit form for the retarded fields

We now obtain more explicit expressions for the retarded fields generated by a particle carrying a dipole moment, in the same way as was done for an electric charge in Section 5.3.5.

We again restrict our attention to the case of an electric dipole; the magnetic case follows by duality. Defining

$$\begin{aligned}
\boldsymbol{E}^d &\equiv \frac{d}{4\pi}\left\{\frac{\boldsymbol{E}_1'^d}{R}+\frac{\boldsymbol{E}_2'^d}{R^2}+\frac{\boldsymbol{E}_3'^d}{R^3}\right\},\\
\boldsymbol{B}^d &\equiv \frac{d}{4\pi}\left\{\frac{\boldsymbol{B}_1'^d}{R}+\frac{\boldsymbol{B}_2'^d}{R^2}+\frac{\boldsymbol{B}_3'^d}{R^3}\right\},
\end{aligned} \tag{5.73}$$

as well as the (unprimed) quantities $\boldsymbol{E}_n^d$ and $\boldsymbol{B}_n^d$:

$$\begin{aligned}
\boldsymbol{E}_n^d &\equiv \frac{\boldsymbol{E}_n'^d}{R^n},\\
\boldsymbol{B}_n^d &\equiv \frac{\boldsymbol{B}_n'^d}{R^n},
\end{aligned}$$

and employing the FitzGerald spin three-vector $\boldsymbol{\sigma}'$ of Section G.4.5,

$$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} - \frac{\gamma}{\gamma+1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}, \tag{5.74}$$

one finds (after an excruciating amount of algebra) that

$$\begin{aligned}
\boldsymbol{E}_1'^d &= \boldsymbol{n}\times\left\{\frac{[\boldsymbol{n}-\boldsymbol{v}]\times\ddot{\boldsymbol{\sigma}}' + \dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}'}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3} + \frac{(\boldsymbol{\sigma}'\cdot\boldsymbol{n})[\boldsymbol{n}-\boldsymbol{v}]\times\ddot{\boldsymbol{v}}}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^4}\right.\\
&\qquad\left. + 3\left(\frac{(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^4} + \frac{(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^5}\right)[\boldsymbol{n}-\boldsymbol{v}]\times\dot{\boldsymbol{v}}\right\},\\
\boldsymbol{E}_2'^d &= \frac{3\,(\dot{\boldsymbol{\sigma}}'\cdot\{[\boldsymbol{n}-\boldsymbol{v}] - \boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v})\})\,[\boldsymbol{n}-\boldsymbol{v}]}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^4} - \frac{\dot{\boldsymbol{\sigma}}'}{\gamma^2[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3}\\
&\quad + \frac{3\,(\boldsymbol{\sigma}'\cdot\{[\boldsymbol{n}-\boldsymbol{v}] - \boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v})\})\,\boldsymbol{n}\times([\boldsymbol{n}-\boldsymbol{v}]\times\dot{\boldsymbol{v}})}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^5}\\
&\quad + \frac{3\,(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\,(\dot{\boldsymbol{v}}\cdot\{[\boldsymbol{n}-\boldsymbol{v}] - \boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v})\})\,[\boldsymbol{n}-\boldsymbol{v}]}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^5}\\
&\quad - \frac{\boldsymbol{\sigma}'\times([\boldsymbol{n}-\boldsymbol{v}]\times\dot{\boldsymbol{v}}) + (\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}'}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3} + \frac{(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})[\boldsymbol{n}-\boldsymbol{v}]}{[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3},\\
\boldsymbol{E}_3'^d &= \frac{3\,(\boldsymbol{\sigma}'\cdot\{[\boldsymbol{n}-\boldsymbol{v}] - \boldsymbol{v}\times(\boldsymbol{n}\times\boldsymbol{v})\})\,[\boldsymbol{n}-\boldsymbol{v}]}{\gamma^2[1-\boldsymbol{v}\cdot\boldsymbol{n}]^5} - \frac{\boldsymbol{\sigma}'}{\gamma^2[1-\boldsymbol{v}\cdot\boldsymbol{n}]^3}.
\end{aligned}$$

In order to obtain these results, it is necessary to use the relations listed in Section G.4 extensively; many of the hundreds of terms that result cancel among themselves; the remainder simplify into the expressions above. The results are further simplified—notationally, at least—by using the convenient quantities $\boldsymbol{n}'$, $\boldsymbol{n}''$ and κ of (5.38), (5.39) and (5.40). One then finds

$$\begin{aligned}
\boldsymbol{E}_1'^d &= \kappa^3\boldsymbol{n}\times\Big\{\boldsymbol{n}'\times\ddot{\boldsymbol{\sigma}}' + \dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}' + \kappa(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{n}'\times\ddot{\boldsymbol{v}}\\
&\qquad + 3\kappa\Big[(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}) + \kappa(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\Big]\boldsymbol{n}'\times\dot{\boldsymbol{v}}\Big\}, && (5.75)\\
\boldsymbol{E}_2'^d &= 3\kappa^4\Big\{(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}'')\boldsymbol{n}' + \kappa(\boldsymbol{\sigma}'\cdot\boldsymbol{n}'')\boldsymbol{n}\times(\boldsymbol{n}'\times\dot{\boldsymbol{v}}) + \kappa(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}'')\boldsymbol{n}'\Big\}\\
&\quad - \kappa^3\Big\{\gamma^{-2}\dot{\boldsymbol{\sigma}}' + \boldsymbol{\sigma}'\times(\boldsymbol{n}'\times\dot{\boldsymbol{v}}) + (\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}' - (\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{n}'\Big\}, && (5.76)\\
\boldsymbol{E}_3'^d &= \gamma^{-2}\kappa^3\Big\{3\kappa^2(\boldsymbol{\sigma}'\cdot\boldsymbol{n}'')\boldsymbol{n}' - \boldsymbol{\sigma}'\Big\}. && (5.77)
\end{aligned}$$

(One should carefully note that the spin derivatives $\dot{\boldsymbol{\sigma}}'$ and $\ddot{\boldsymbol{\sigma}}'$ appearing here are the lab-time derivatives *of the FitzGerald spin definition (5.74)*, in accordance to the precedence-of-operations rules of Section A.3.8; they are *not* the "FitzGerald–Lorentz contractions" of the standard spin derivatives $\dot{\boldsymbol{\sigma}}$ and $\ddot{\boldsymbol{\sigma}}$.)

To compute the magnetic field expressions $\boldsymbol{B}'^{d}_{n}$, we can again use the properties (5.34) and (5.35). Inspection of (5.70) and (5.72) shows that $\boldsymbol{B}'^{d}_{1}$ satisfies (5.35) and $\boldsymbol{B}'^{d}_{3}$ satisfies (5.34). However, (5.71) is more problematical, mixing terms of both types, and indeed having one term of neither type. If one inspects the detailed expressions carefully, and performs some algebraic shuffling, one finds that $\boldsymbol{B}'^{d}_{2}$ can ultimately be written as the sum of a term like (5.35), and two additional terms. The complete results are then

$$\boldsymbol{B}'^{d}_{1} = \boldsymbol{n}\times\boldsymbol{E}'^{d}_{1}, \tag{5.78}$$

$$\boldsymbol{B}'^{d}_{2} = \boldsymbol{n}\times\boldsymbol{E}'^{d}_{2} + \kappa^{2}\boldsymbol{\sigma}'\times\dot{\boldsymbol{v}} + \kappa^{3}\boldsymbol{n}'\times\Big[\boldsymbol{n}\times(\boldsymbol{\sigma}'\times\dot{\boldsymbol{v}})\Big], \tag{5.79}$$

$$\boldsymbol{B}'^{d}_{3} = \boldsymbol{v}\times\boldsymbol{E}'^{d}_{3}. \tag{5.80}$$

The corresponding results for a point particle carrying a *magnetic* dipole moment are, of course, simply the electromagnetic duals of these expressions—plus an extra Maxwell term at the position of the particle, which will be discussed in Section 5.5.

5.4.7 Simplicity of the author's expressions

The equations (5.75), (5.76), (5.77), (5.78), (5.79) and (5.80) are the author's final results for the retarded fields from an electric dipole. They are, arguably, as simple as one can get.

An illustration of this assertion is the fact that, even though the expressions above are in fact true for an *arbitrary velocity* $\boldsymbol{v}$ of the particle, the quantity $\boldsymbol{v}$ itself only appears *once*, in one of the terms of $\boldsymbol{E}'^{d}_{2}$; and the corresponding gamma factor $\gamma(\boldsymbol{v})$ only appears twice, in the form γ^{-2}. The velocity-dependencies of the results are, in effect, almost entirely "encapsulated" in the four introduced quantities $\boldsymbol{\sigma}'$, $\boldsymbol{n}'$, $\boldsymbol{n}''$ and κ. Thus, even if one goes to the *instantaneous rest frame* of the particle, one cannot improve on the simplicity of the expressions much further: all that happens is that the aforementioned term in $\boldsymbol{E}_2^{\prime d}$ disappears; the factors κ and γ disappear, being trivially equal to unity; the dashes are dropped on $\boldsymbol{n}'$, $\boldsymbol{n}''$, $\boldsymbol{\sigma}'$ and $\dot{\boldsymbol{\sigma}}'$; and we must make the simple substitution

$$\ddot{\boldsymbol{\sigma}}' \longrightarrow \ddot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}}.$$

(This last subtlety comes about because, in computing two temporal derivatives of the FitzGerald spin definition (5.74), a term survives in which both factors of $\boldsymbol{v}$ have been differentiated into $\dot{\boldsymbol{v}}$; it has a coëfficient of unity, rather than the one-half implied by $\gamma/(\gamma+1)$, because of the quadratic dependence on $\boldsymbol{v}$.)

It should, at this point, be noted that the expressions (5.75), (5.76), (5.77), (5.78), (5.79) and (5.80) are *only* as simple as they are because they use the *FitzGerald three-spin*, $\boldsymbol{\sigma}'$, and its lab-time derivatives. If expressed in terms of the *standard* three-spin, $\boldsymbol{\sigma}$, and its derivatives, the results are much more complicated: there are many more terms, most of which contain explicit factors of $\boldsymbol{v}$, γ and $(\gamma+1)$. Indeed, the author only *invented* the FitzGerald three-spin, originally, in an attempt to obtain just such a simplification. (The quantity $\boldsymbol{\sigma}'$ was actually introduced by the author, previously, as one of the many "convenient quantities" useful for the expanded-out dipole equations of motion (*see* Appendix F, and [62, 63, 64, 65]), but its usefulness as a general analytical tool was not realised at the time.) The basic reason *why* the FitzGerald three-spin yields simpler expressions than the standard three-spin can be gleaned from the expressions listed in Section G.4: in terms of $\boldsymbol{\sigma}'$, the components of the four-spin Σ have the simple coëfficients 1 and γ^2; but in terms of $\boldsymbol{\sigma}$, there is, instead of γ^2, a coëfficient of $\gamma^2/(\gamma+1)$;

under differentiation, the latter expression generates even more complicated coëfficients (compare the lengthy Σ-derivative expressions in Section G.4.7 and Section G.4.8 to their much simpler counterparts in Section G.4.9).

Returning, now, to the expressions (5.75), (5.76), (5.77), (5.78), (5.79) and (5.80), one can immediately recognise familiar faces among the results. Most obviously, the $1/R^3$ electric field (5.77) contains the *static* dipole field expression most transparently:

$$\boldsymbol{E}_3'^{d}\Big|_{v=0} = 3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \boldsymbol{\sigma};$$

the full velocity dependence of the result (5.77) merely incorporates the Lorentz transformation of these static fields by the velocity $\boldsymbol{v}$. (The factor of $1/r^3 \equiv 1/R^3$ is of course encapsulated in the notational definitions of (5.73).) The corresponding $1/R^3$ *magnetic* field of (5.80) is, as one would expect, simply the cross-product of the three-velocity and the corresponding electric field (5.77). (That one would expect this result, in advance, is recognised by noting that the Lorentz-boosted *electric charge* static fields (5.42) and (5.44) themselves possess this property; if one considers a static electric dipole to consist of two spatially separated electric charges in the rest frame of the particle,—which themselves will be stationary, if the dipole moment is not precessing,—then the transformation properties of the dipolar static fields under Lorentz boosts *must* be identical to those of the electric charge.)

Likewise, the $1/R$ electric and magnetic fields of (5.75) and (5.78) are, *manifestly*, perpendicular to both the normal vector $\boldsymbol{n}$, and to each other,—and of the same magnitude,—as must of course be the case for an electromagnetic radiation field. (Indeed, this follows from the fact that the covariant field expressions (5.70), (5.71) and (5.70) of Section 5.4.5 are in accord with the *Goldberg–Kerr theorem* [93]).

The *particular* terms contained in the results (5.75), (5.76) and (5.79) would not, of course, be very familiar to many readers. But the author suggests that this unfamiliarity is due solely to one's simply not having been introduced to them before, *not* because they exhibit any inherent complexity. In fact, if one examines the terms in (5.75) and (5.76) particularly, one finds that, really, they are pretty well what one would *expect* to find, simply on dimensional and consistency grounds. For starters, one knows that, since the fields are being generated by a electric dipole moment $\boldsymbol{d}$—represented here by $d\,\boldsymbol{\sigma}$,—each term in the field expressions *must* contain one, and *only* one, factor of $\boldsymbol{\sigma}'$, or one of its time-derivatives. Furthermore, one knows that, simply on dimensional grounds, the terms in the $1/R^2$ and $1/R$ fields *must* contain a total of one and two overdots, respectively—since the three-vectors $\boldsymbol{v}$, $\boldsymbol{\sigma}'$, $\boldsymbol{n}$, and their cohorts, are all dimensionless. One is then simply left with the therapeutic pastime of slotting together the available three-vectors, and the available number of overdots, into terms of the correct parity (axial vectors for the terms in braces in (5.75); polar vectors for the terms in (5.76)). As already noted, the three-vector $\boldsymbol{v}$ is itself only used once; *a priori*, one would not expect such a dramatic simplification of the gameplay. From here, one must then simply sprinkle the scalar factors γ and κ (and the numerical coëfficient 3) around liberally, for one to end up with expressions looking like (5.75) and (5.76).

In any case, regardless of their æsthetic merit, the author believes the expressions (5.75), (5.76), (5.77), (5.78), (5.79) and (5.80) to be most definitely correct, so one will now have to live with them, regardless of one's opinions of them.

5.5 The static fields

We now wind back the throttles on our analytical engines somewhat, to provide an overview of the standard textbook problem of obtaining the *static* fields of a point particle carrying electric charge and electric and magnetic dipole moments; and, in particular, the behaviour of these fields *at the position of the particle*.

Because the field expressions diverge at this point, we "regularise" the physical situation by considering not a point particle, which possesses the source densities

$$a_\rho(\boldsymbol{r}) = a\delta(\boldsymbol{r})$$

(where $a = q, d, \mu$), but rather we consider a particle which is a three-sphere of radius ε, over which the electromagnetic moments are *uniformly distributed*. (The relativistic subtleties of such a model when the particle in question is *accelerated* are described in detail in Chapter 3.) At the end of the calculations, one may, if one wishes, shrink the body back to a point, by taking the limit $\varepsilon \to 0$; but, in fact, we shall, in Chapter 6, need a number of results computed here *before* the point limit is taken, *i.e.*, those results for which ε is kept finite.

Whilst there is nothing new in this section, over what is contained in any undergraduate course in electrostatics and magnetostatics, we include here the derivation to highlight those aspects and results that are of importance to the radiation reaction calculations of Chapter 6.

5.5.1 The fields from an electric charge

The static field of a finite sphere of uniform *electric charge* density is simple and well known; since Poisson's equation covers electrostatics and Newtonian gravitostatics equivalently, the electric field of a uniformly charged sphere is the same as the Newtonian gravitational field from a spherical volume of uniform mass density [158, Prop. LXXIII, Book I]:

$$\boldsymbol{E}^q(\boldsymbol{r}) = \begin{cases} \dfrac{qr\boldsymbol{n}}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \dfrac{q\boldsymbol{n}}{4\pi r^2}, & r > \varepsilon, \end{cases} \tag{5.81}$$

where q is the total charge of the sphere, and where, for simplicity, we place the centre of the sphere at the origin of coördinates, $\boldsymbol{r} = \boldsymbol{0}$.

One can very simply verifiy that the stated result (5.81) is correct for a uniformly charged sphere, by taking its divergence and curl. Noting that the expression (5.81) is continuous everywhere (in particular, there are no abrupt jumps anywhere on the surface of the sphere, where the functional form of the result changes), and making use of the identities of Section B.3, one finds

$$\nabla\cdot\boldsymbol{E}^q(\boldsymbol{r}) = \begin{cases} \dfrac{3q}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \\ 0, & r > \varepsilon, \end{cases} \tag{5.82}$$

which is of course recognised as the correct density of electric charge, when one recalls that the volume of the sphere is $4\pi\varepsilon^3/3$. The curl of (5.81) is, of course, zero.

(The boundary condition implicitly assumed in obtaining (5.81) as the integral of (5.82) is, of course, that the fields vanish at spatial infinity.)

5.5.2 The fields from an electric dipole

For a spherical body throughout which an *electric dipole moment density* is uniformly distributed, we may obtain the static field most simply by noting that each *elementary* constituent dipole is equivalent to two equal and opposite electric charges placed infinitesimally close together, aligned in the direction of $\boldsymbol{d}$; and that all of the $\boldsymbol{d}$ three-vectors of the constituent dipoles are coherently aligned, when the body as a whole is static. This means that we may simply superpose the fields of the constituents according to the relation valid for each:

$$\boldsymbol{E}^d(\boldsymbol{r}) = -\frac{1}{q}(\boldsymbol{d}\cdot\nabla)\boldsymbol{E}^q(\boldsymbol{r}).$$

(The minus sign appears because by separating the charges by an infinitesimal amount we are actually shifting the *origin* of coördinates, *not* the position where we measure the field.) By obtaining the gradient of the charge expression (5.81) in the direction of $\boldsymbol{d}$, by direct calculation, we thus find that

$$\boldsymbol{E}^d(\boldsymbol{r}) = \begin{cases} -\dfrac{\boldsymbol{d}}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \dfrac{3(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n} - \boldsymbol{d}}{4\pi r^3}, & r > \varepsilon. \end{cases} \tag{5.83}$$

5.5.3 The fields from a magnetic dipole

Now consider a spherical body over which a *magnetic* dipole moment density is uniformly distributed. Naïvely, one might expect the result to be simply the electromagnetic dual of the electric dipole result, (5.83). However, one must remember that the magnetic and electric dipole results should only be expected to be electromagnetically dual *away from sources*—since the inhomogeneous Maxwell equations do *not*, of course, possess duality symmetry at all (since magnetic charge is not allowed). By assuming the magnetic dipole result to be the dual of the electric, one would be assuming that it could, by the above, be obtained by considering infinitesimally separated *magnetic* charges; this is plainly unphysical.

The problem may be seen by computing the divergence of (5.83). Let us assume, for simplicity, that $\boldsymbol{d}$ is in the z-direction. At the diametrically opposite points on the surface of the spherical body along the z-axis, we find

$$\nabla\cdot\boldsymbol{E}^d\Big|_{x=y=0,\, z=\pm\varepsilon} = \partial_z E_z^d\Big|_{x=y=0,\, z=\pm\varepsilon} = \pm\frac{3}{4\pi\varepsilon^3}\delta(z-\varepsilon),$$

due to the step-function change in E_z^d in crossing the edge of the body. On the other hand, in the x–y plane, we find

$$\nabla\cdot\boldsymbol{E}^d\Big|_{x^2+y^2=\varepsilon^2,\, z=0} = 0,$$

since the field (5.83) matches smoothly across this circular boundary. We thus find the familiar picture of a spherical, uniformly electrically polarised volume as being in all respects equivalent to a sphere with a *surface charge* *sheet* wrapping the sphere, with the charge density being positive in the direction of $\boldsymbol{d}$, negative antiparallel to $\boldsymbol{d}$, and zero perpendicular to $\boldsymbol{d}$ (more specifically, varying as $\sin\theta$, where $\theta = \pi/2$ in the direction of $\boldsymbol{d}$, and $-\pi/2$ antiparallel to $\boldsymbol{d}$).

But this cannot, of course, be generalised to the magnetic case, since we would then have a *magnetic charge sheet* wrapping the spherical volume, which is, as noted, fundamentally disallowed by the Maxwell equations. However, we *do* know that the field *external* to the spherical source volume will be the dual of the electric case. Let us then write

$$\boldsymbol{B}^{\mu}(\boldsymbol{r}) = \begin{cases} -\dfrac{\boldsymbol{\mu}}{4\pi\varepsilon^3} + \boldsymbol{B}^{\mu}_M(\boldsymbol{r}), & r < \varepsilon, \\ \dfrac{3(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n} - \boldsymbol{\mu}}{4\pi r^3}, & r > \varepsilon, \end{cases}$$

where the extra *magnetic Maxwell field*, $\boldsymbol{B}^{\mu}_M(\boldsymbol{r})$, will be constructed so as to repair the breach of Maxwell's equations that would otherwise occur. Now, since there is no *net* electric current in the *interior* of the sphere (since the effects of any current loop at one position would be cancelled out by that of its neighbour—except on the surface of the sphere—due to the fact that the polarisation is uniform), and since everything is electromagnetostatic, Maxwell's equations require

$$\nabla\cdot\boldsymbol{B}^{\mu}_M = 0$$

and

$$\nabla\times\boldsymbol{B}^{\mu}_M = \boldsymbol{0};$$

hence, $\boldsymbol{B}^{\mu}_M$ must be constant throughout the sphere. We must therefore only determine its magnitude and direction. Clearly, the direction of $\boldsymbol{B}^{\mu}_M$ singles out one particular direction in space; since the only special direction we have available to us is $\boldsymbol{\mu}$, we must have

$$\boldsymbol{B}^{\mu}_M(\boldsymbol{r}) = B^{\mu}_M\hat{\boldsymbol{\mu}}.$$

We now note that the non-zero electric charge density of (5.83) arises solely because the electric field at $x = y = 0$, $z = \pm\varepsilon$ has a value $2d/4\pi\varepsilon^3$ outside the sphere, but $-d/4\pi\varepsilon^3$ inside; if the internal field were increased to $2d/4\pi\varepsilon^3$ (by adding an amount $3d/4\pi\varepsilon^3$), then the divergence would vanish. We thus conclude that the extra Maxwell field should be

$$\boldsymbol{B}^{\mu}_{M}(\boldsymbol{r}) = \begin{cases} \dfrac{3\boldsymbol{\mu}}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \\ \boldsymbol{0}, & r > \varepsilon, \end{cases} \tag{5.84}$$

and hence the total magnetic field is

$$\boldsymbol{B}^{\mu}(\boldsymbol{r}) = \begin{cases} +\dfrac{2\boldsymbol{\mu}}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \dfrac{3(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n} - \boldsymbol{\mu}}{4\pi r^3}, & r > \varepsilon. \end{cases} \tag{5.85}$$

To verify that the result (5.85) is, indeed, physically acceptable, we must first recall that the smooth matching of $\boldsymbol{E}^d$ around the surface of the sphere in the x–y plane will now *not* be the case, since the extra internal field $3\boldsymbol{\mu}/4\pi\varepsilon^3$ of (5.84) applies there also. However, this step change in the field does *not* add any magnetic charge, since the step is in the z-component of $\boldsymbol{B}$, but occurs in traversing the boundary in the x and/or y directions; there is no contribution to the divergence of $\boldsymbol{B}$. Rather, we find a delta-function contribution to the *curl* of $\boldsymbol{B}$—and, hence, find the familiar picture of a uniformly magnetically polarised sphere as being in all fundamental aspects equivalent to a sphere with a *current sheet* wrapped around its surface, where the current density has a $\cos\theta$ dependence.

5.5.4 Point dipole fields, and Dirac delta functions

We now clarify an aspect of the static dipole fields, derived in the previous sections, that is often muddied somewhat in textbook descriptions: namely, a correct description of the fields of *point* particles, by making use of the three-dimensional Dirac delta function $\delta(\boldsymbol{r})$ to represent the field at the position of the particle.

The motivation behind such an approach is easily appreciated, if one considers (5.83) or (5.85) in somewhat more detail: The field strength *inside* the spherical source volume is constant, and diverges like ε^{-3}; but on the other hand, the *volume* of the sphere is of order ε^{+3}; thus, *the product of the two is independent of* ε. Explicitly, we find

$$\int_{r<\varepsilon} d^3r\, \boldsymbol{E}^d(\boldsymbol{r}) = -\frac{1}{3}\boldsymbol{d}, \tag{5.86}$$

$$\int_{r<\varepsilon} d^3r\, \boldsymbol{B}^\mu(\boldsymbol{r}) = +\frac{2}{3}\boldsymbol{\mu}, \tag{5.87}$$

where the difference between (5.86) and (5.87) is attributable to the extra Maxwell magnetic field (5.84). One is therefore naturally led to think of a three-dimensional *Dirac delta function* as a convenient description for the internal fields, since it also has the property of diverging at the origin, while having a constant, finite volume integral.

The problem arises when one tries to formulate this property mathematially. *One* approach is to simply "rip the guts" out of the dipolar inverse-cube function

$$\frac{3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\boldsymbol{\sigma}}{4\pi r^3}, \tag{5.88}$$

—namely, one simply defines a new function that has, by decree, the value *zero* at $\boldsymbol{r}=\boldsymbol{0}$. Let as refer to this "gutted" function as the *principal value* of the function (5.88), and denote it by prefixing the function by the symbol '$\mathcal{P}$'. Such an operation may be carried out by considering a limiting procedure akin to that used for the spherical body above: one may define

$$\mathcal{P}\frac{3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\boldsymbol{\sigma}}{4\pi r^3} \equiv \lim_{\varepsilon\to 0}\begin{cases} \boldsymbol{0}, & r<\varepsilon, \\ \dfrac{3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\boldsymbol{\sigma}}{4\pi r^3}, & r>\varepsilon; \end{cases} \tag{5.89}$$

clearly, for any ε, this parametrisation of the principal value function vanishes around $\boldsymbol{r} = \boldsymbol{0}$, and hence also does so in the point limit. In terms of (5.89), the point particle dipole results, from (5.83), (5.85), (5.86) and (5.87), can be written

$$\begin{aligned} \boldsymbol{E}^{d}_{\text{point}}(\boldsymbol{r}) &= \mathcal{P}\frac{3(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n} - \boldsymbol{d}}{4\pi r^3} - \frac{1}{3}\boldsymbol{d}\,\delta(\boldsymbol{r}), \\ \boldsymbol{B}^{\mu}_{\text{point}}(\boldsymbol{r}) &= \mathcal{P}\frac{3(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n} - \boldsymbol{\mu}}{4\pi r^3} + \frac{2}{3}\boldsymbol{\mu}\,\delta(\boldsymbol{r}). \end{aligned} \tag{5.90}$$

There is, however, somewhat of a philosophical objection to the "gutting" procedure used to define the principal value function, expression (5.89), which can be understood as follows: The *electric* dipole result for a small sphere, equation (5.83), was obtained by considering the superposition of a continuous volume of infinitesimal dipoles. Unlike the magnetic case, *no* extra fields were added, by hand, to the result. Thus, one could, conceptually, consider shrinking the dipole generating the field (5.83) *itself* to a point—and then use *this* as a constituent in an a *larger* finite sphere dipole; and so on, *ad infinitum*. It therefore seems arguably natural to simply *define* the dipolar inverse-cube function to be the limit of the result (5.83), in the point limit. Let us refer to this as the *regularised value* of the function (5.88), and denote it by prefixing the function by the symbol '$\mathcal{R}$':

$$\mathcal{R}\frac{3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \boldsymbol{\sigma}}{4\pi r^3} \equiv \lim_{\varepsilon\to 0}\begin{cases} -\dfrac{\boldsymbol{\sigma}}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \dfrac{3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \boldsymbol{\sigma}}{4\pi r^3}, & r > \varepsilon; \end{cases} \tag{5.91}$$

The reason for christening it so is that, regardless of which way one "regularises" the function (5.88)—in other words, modifying its definition so that it remains finite, rather than mathematically divergent,—one invariably finds a result equivalent to (5.83), in the point limit, rather than that of the principal value (5.89). (*See* Chapter 6 for an example of this.)

Regardless of the philosophical arguments one way or the other, it is nevertheless true that the principal value definition, (5.89), and the regularised value definition, (5.89), are equally valid mathematical choices, *provided one is consistent about which definition one uses*.

In terms of the regularised function (5.91), the static electric and magnetic dipole fields for pointlike particles can thus be written

$$\begin{aligned} \boldsymbol{E}^d_{\text{point}}(\boldsymbol{r}) &= \mathcal{R}\frac{3(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n}-\boldsymbol{d}}{4\pi r^3}, \\ \boldsymbol{B}^\mu_{\text{point}}(\boldsymbol{r}) &= \mathcal{R}\frac{3(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}}{4\pi r^3} + \boldsymbol{\mu}\,\delta(\boldsymbol{r}). \end{aligned} \tag{5.92}$$

The conceptual advantage in using this method of expression, for our present purposes, is clear: the electric dipole field—obtainable directly from that of the electric monopole (charge) field—is simply the naïve gradient of the latter in the direction of $\boldsymbol{d}$; the *magnetic* dipole field, on the other hand—which requires the addition of the extra Maxwell contribution $\boldsymbol{B}_M(\boldsymbol{r})$,—exhibits this extra contibution *manifestly* in (5.92):

$$\boldsymbol{B}^\mu_M(\boldsymbol{r}) = \boldsymbol{\mu}\,\delta(\boldsymbol{r}). \tag{5.93}$$

Again, regardless of one's choice, one must simply ensure that one sticks to it consistently.

5.5.5 Point particle source terms

For convenience, we list here the electric *source terms* $\rho(\boldsymbol{r})$ and $\boldsymbol{J}(\boldsymbol{r})$ that appear in the Maxwell equations corresponding to a point electric charge, point electric dipole moment and point magnetic dipole moment.

For a point electric charge, we clearly have

$$\begin{aligned} \rho_q(\boldsymbol{r}) &= q\delta(\boldsymbol{r}), \\ \boldsymbol{J}_q(\boldsymbol{r}) &= \boldsymbol{0}. \end{aligned}$$

For a point electric dipole moment, we note that it is fundamentally formed by the infinitesimal separation of positive and negative charge; we thus have

$$\rho_d(\boldsymbol{r}) = -(\boldsymbol{d}\cdot\nabla)\delta(\boldsymbol{r}),$$
$$\boldsymbol{J}_d(\boldsymbol{r}) = \boldsymbol{0}.$$

(The sign in ρ is negative because the derivative $d_t\delta(t)$ of the Dirac delta function $\delta(t)$ is *negative* for $t > 0$ and *positive* for $t < 0$; an infinitesimal dipole has, by convention, the senses of its charges reversed from this.)

Finally, for a point magnetic dipole moment, we must be a little more careful. The *net charge density* for such an object *vanishes*:

$$\rho_\mu(\boldsymbol{r}) = 0.$$

The dipole moment of course arises from the *circulating electric current* at the position of the dipole; clearly, we then have

$$\boldsymbol{J}_\mu(\boldsymbol{r}) = -\boldsymbol{\mu}\times\nabla\delta(\boldsymbol{r}).$$

5.5.6 Mechanical properties of the self-fields

We now consider the mechanical energy, mechanical momentum and mechanical angular momentum stored in the static fields of the uniform-density spherical body. As pointed out by Einstein [76], the electromagnetic mechanical energy of the static fields will contribute to the *mass* of the body. A non-vanishing total electromagnetic mechanical momentum of the static fields would violate relativistic principles. The total electromagnetic mechanical angular momentum, on the other hand, may be interpreted as a contribution to the mechanical angular momentum of the body in question (or "mechanical spin"), in the same way as the mechanical energy contributes to the mass.

We assume that the electric and magnetic dipole moments of the body are, in fact, *parallel*, in the following discussion; this has not been assumed in the previous sections of this chapter, but is, of course, the case for spin-half particles.

5.5.7 Mechanical field expressions

The densities of the electromagnetic mechanical energy, mechanical momentum and mechanical angular momentum are given by [113, Sect. 6.8]

$$W_\rho(\boldsymbol{r}) = \frac{1}{2}\Big\{\boldsymbol{E}^2(\boldsymbol{r}) + \boldsymbol{B}^2(\boldsymbol{r})\Big\}, \tag{5.94}$$

$$\boldsymbol{p}_\rho(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r}), \tag{5.95}$$

$$\boldsymbol{s}_\rho(\boldsymbol{r}) = \boldsymbol{r}\times\boldsymbol{p}_\rho(\boldsymbol{r}). \tag{5.96}$$

It should be noted that, due to the nonlinearity of these relations, we must consider *all* of the possible combinations of multiple moments explicitly: we *cannot* superpose the results for the moments considered individually.

5.5.8 Electric charge mechanical field properties

If the body possesses electric charge only, then, since it generates no static magnetic field, relations (5.95) and (5.96) show that its mechanical self-momentum and mechanical angular self-momentum vanish identically. From (5.81) and (5.94), its mechanical self-energy is given by

$$\begin{aligned} m^q_{\text{e.m.}} &= \frac{1}{2}\int_{r<\varepsilon} d^3r\,\frac{q^2r^2}{16\pi^2\varepsilon^6} + \frac{1}{2}\int_{r>\varepsilon} d^3r\,\frac{q^2}{16\pi^2r^4} \\ &= \frac{3q^2}{20\pi\varepsilon} \\ &\equiv \frac{1}{2}q^2\eta_1, \end{aligned} \tag{5.97}$$

where the convenient constant η_1 will play a significant rôle in Chapter 6. This contribution to (or "renormalisation of") the mass is, for a point particle, infinite; in the classical theory, one must assume that there are other (non-electromagnetic) contributions to the mass of the body that approach minus infinity in the point limit, leaving a finite mass overall.

5.5.9 Electric dipole mechanical field properties

For a body possessing an electric dipole moment only, we again find that the mechanical self-momentum and mechanical angular self-momentum vanish identically. From (5.83) and (5.94), its mechanical self-energy is given by

$$\begin{aligned} m^d_{\text{e.m.}} &= \frac{1}{2}\int_{r<\varepsilon} d^3r\,\frac{d^2}{16\pi^2\varepsilon^6} + \frac{1}{2}\int_{r>\varepsilon} d^3r\,\frac{\{3(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n}-\boldsymbol{d}\}^2}{16\pi^2 r^6} \\ &= \frac{d^2}{8\pi\varepsilon^3} \\ &\equiv \frac{1}{2}d^2\eta_3', \end{aligned} \tag{5.98}$$

where the convenient constant η_3' will again play a significant rôle in Chapter 6.

Again, (5.98) represents an infinite renormalisation of the mass of the body in the point limit.

5.5.10 Magnetic dipole mechanical field properties

For a body possessing a magnetic dipole moment only, we again find vanishing mechanical self-momentum and mechanical angular self-momentum. Of importance is the fact that the mechanical self-energy is *not* equivalent to the electric dipole case, since the contribution from the internal field is four times as large (since the magnitude of the internal field is twice that of the electric case, due to the addition of the extra Maxwell magnetic field $\boldsymbol{B}^{\mu}_{M}$).

From (5.85) and (5.94), one finds

$$\begin{aligned} m^{\mu}_{\text{e.m.}} &= \frac{1}{2}\int_{r<\varepsilon} d^3r\,\frac{4\mu^2}{16\pi^2\varepsilon^6} + \frac{1}{2}\int_{r>\varepsilon} d^3r\,\frac{\{3(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}\}^2}{16\pi^2 r^6} \\ &= \frac{\mu^2}{4\pi\varepsilon^3} \\ &= \mu^2\eta'_3; \end{aligned} \tag{5.99}$$

in other words, the contribution to the mass is *twice* that which one would naïvely assume if one did not consider the extra field $\boldsymbol{B}^{\mu}_{M}$ of (5.84).

The mass contribution is, of course, again an infinite renormalisation in the point limit.

We now consider particles which carry multiple electromagnetic moments.

5.5.11 Particle with electric and magnetic moments

For the case of a body possessing both an electric and a magnetic dipole moment, we note that the mechanical self-energy density will be simply the sum of the individual densities (since the static electric dipole generates no magnetic field, and the static magnetic dipole generates no electric field). The mechanical self-momentum density will at all points in space vanish, since the electric field of the electric dipole and the magnetic field of the magnetic dipole are, at all points in space, in the same direction (even though the internal fields have a different overall magnitude and sign), and hence their cross product vanishes. This also implies that the mechanical angular self-momentum density vanishes at all points in space.

5.5.12 Charged magnetic dipole

For the case of a body carrying both electric charge and a magnetic dipole moment, we note, again, that the mechanical self-energy density is simply the sum of those of the charge and the magnetic dipole individually, because the former generates solely an electric field, and the latter solely a magnetic field. However, the mechanical self-momentum density as a function of $\boldsymbol{r}$ does *not* vanish: it is given by

$$\boldsymbol{p}_\rho^{q\mu}(\boldsymbol{r}) = \begin{cases} +\dfrac{2qr\boldsymbol{n}\times\boldsymbol{\mu}}{(4\pi)^2\varepsilon^6}, & r<\varepsilon, \\ -\dfrac{q\boldsymbol{n}\times\boldsymbol{\mu}}{(4\pi)^2 r^5}, & r>\varepsilon. \end{cases}$$

Now, these expressions are both odd in $\boldsymbol{r}$, and hence vanish when integrated over all space; there is no net mechanical self-momentum. However, when we cross the vector $\boldsymbol{r}$ into $\boldsymbol{p}_\rho^{q\mu}(\boldsymbol{r})$, to compute $\boldsymbol{s}_\rho^{q\mu}(\boldsymbol{r})$, we find

$$\boldsymbol{s}_\rho^{q\mu}(\boldsymbol{r}) = \begin{cases} +\dfrac{2qr^2\{(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}\}}{(4\pi)^2\varepsilon^6}, & r<\varepsilon, \\ -\dfrac{q\{(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}\}}{(4\pi)^2 r^4}, & r>\varepsilon, \end{cases}$$

which are of course *even* in $\boldsymbol{r}$. Upon integration, we thus find that

$$\begin{aligned} \boldsymbol{s}_{\text{e.m.}}^{q\mu} &= \int_{r<\varepsilon} d^3r\,\frac{2qr^2\{(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}\}}{(4\pi)^2\varepsilon^6} - \int_{r>\varepsilon} d^3r\,\frac{q\{(\boldsymbol{n}\cdot\boldsymbol{\mu})\boldsymbol{n}-\boldsymbol{\mu}\}}{(4\pi)^2 r^4} \\ &= \frac{q\boldsymbol{\mu}}{10\pi\varepsilon} \\ &= \frac{1}{3}q\boldsymbol{\mu}\eta_1. \end{aligned} \tag{5.100}$$

Thus, a charged magnetic dipole has an electromagnetic contribution to its mechanical spin angular momentum, in the direction of the magnetic moment (which, for spin-half particles, will be in the direction of its bare mechanical spin), of value $q\mu\eta_1/3$. This repesents *an infinite renormalisation of the spin of the body*.

5.5.13 Charged electric dipole

For a body possessing both electric charge and electric dipole moment, there is again only an electric field generated, and hence the mechanical self-momentum and mechanical angular self-momentum vanish trivially. The mechanical self-energy density, on the other hand, is given by

$$\begin{aligned} W_\rho^{qd}(\boldsymbol{r}) &= \frac{1}{2}\boldsymbol{E}^2(\boldsymbol{r}) \\ &= \frac{1}{2}\Big\{\boldsymbol{E}^q(\boldsymbol{r}) + \boldsymbol{E}^d(\boldsymbol{r})\Big\}^2 \\ &= \frac{1}{2}\boldsymbol{E}^{q\,2}(\boldsymbol{r}) + \frac{1}{2}\boldsymbol{E}^{d\,2}(\boldsymbol{r}) + \boldsymbol{E}^q(\boldsymbol{r})\cdot\boldsymbol{E}^d(\boldsymbol{r}). \end{aligned} \tag{5.101}$$

Now, the first two terms on the last line of (5.101) are just the mechanical self-energy densities for the electric charge and the electric dipole moment when they are considered *singly*, as was done in Sections 5.5.8 and 5.5.9; we need not analyse these expressions anew. However, the *final*, "cross" term in (5.101), arising through the nonlinearity of (5.94), represents an *interference between the charge and dipole fields*. Let us concentrate on this interference term, and denote it by $\varDelta W_\rho^{qd}(\boldsymbol{r})$:

$$\varDelta W_\rho^{qd}(\boldsymbol{r}) \equiv \boldsymbol{E}^q(\boldsymbol{r})\cdot\boldsymbol{E}^d(\boldsymbol{r}). \tag{5.102}$$

Using (5.81) and (5.83), we find

$$\varDelta W_\rho^{qd}(\boldsymbol{r}) = \begin{cases} -\dfrac{qr(\boldsymbol{n}\cdot\boldsymbol{d})}{(4\pi)^2\varepsilon^6}, & r < \varepsilon, \\[2ex] +\dfrac{2q(\boldsymbol{n}\cdot\boldsymbol{d})}{(4\pi)^2 r^5}, & r > \varepsilon. \end{cases} \tag{5.103}$$

Now, both of the expressions in (5.103) are odd in $\boldsymbol{n}$; thus, when we integrate $\varDelta W_\rho^{qd}(\boldsymbol{r})$ over all space, we obtain no net contribution to the mechanical self-energy of the charged electric dipole.

It would be convenient if that were the end of the story. However, the alert reader will realise that the spatial asymmetry of the expression (5.103) has a subtle, almost sinister, by-product: *the mechanical centre of energy of the self-field does not coïncide with the centre of charge and polarisation.*

The position of the mechanical centreof energy, $\boldsymbol{z}_{\rm CoE}$, is, of course, simply the energy-weighted expectation value of the position operator $\boldsymbol{r}$:

$$\boldsymbol{z}_{\rm CoE} \equiv \frac{\int d^3r\, \boldsymbol{r} W_\rho(\boldsymbol{r})}{\int d^3r\, W_\rho(\boldsymbol{r})}. \tag{5.104}$$

If one goes back and examines the expressions carefully, one finds that the mechanical self-energy densities considered in previous sections were all *even in* $\boldsymbol{r}$ (automatically ensured, of course, if the density is simply a sum of squared fields, as it was in those sections); application of (5.104) then trivially yields $\boldsymbol{z}_{\rm CoE} = \boldsymbol{0}$, the centre of our sphere. But since the expression (5.103) for the interference contribution $\Delta W_\rho^{qd}(\boldsymbol{r})$ is *odd* in $\boldsymbol{r}$, the subsequent contribution to (5.104) then yields a *non-zero* result; specifically,

$$\begin{aligned}(m^q_{\rm e.m.} + m^d_{\rm e.m.})\boldsymbol{z}^{qd}_{\rm CoE} &= -\int_{r<\varepsilon} d^3r\, \frac{qr^2(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n}}{(4\pi)^2\varepsilon^6} + \int_{r>\varepsilon} d^3r\, \frac{2q(\boldsymbol{n}\cdot\boldsymbol{d})\boldsymbol{n}}{(4\pi)^2 r^4} \\ &= \frac{3q\boldsymbol{d}}{20\pi\varepsilon} \\ &= \frac{1}{2} q\boldsymbol{d}\eta_1. \end{aligned} \tag{5.105}$$

We can see, from the result (5.105), that the mechanical centre of energy of the uniform-density spherical charged electric dipole is *offset* from the centre of the sphere, in the direction of the electric dipole moment $\boldsymbol{d}$. On the other hand, this offset *vanishes* in the point limit, since $m^d_{\rm e.m.}$ is of order ε^{-3}, whereas the right-hand side of (5.105) is only of order ε^{-1}.

This effect has serious ramifications for the radiation reaction calculations of Chapter 6 (which uses just such a model to regularise the pointlike charged electric dipole); we clearly need to add a "mass dipole" to the system to return the mechanical cenre of energy to the centre of the spherical body. Note that, since the radiation reaction calculation proceeds on the basis of *finite* ε, with the point limit only being taken at the *end* of the calculations, we *cannot* simply rely on the vanishing of the centre-of-energy offset in the point limit for the radiation reaction calculations themselves.

5.5.14 Particle with all three moments

As noted in Secion 5.5.7, the nonlinearity of the mechanical field density expressions (5.94), (5.95) and (5.96) means that we have to consider separately the cases of *all possible combinations* of the three electromagnetic moments of interest to us (charge, electric dipole moment, and magnetic dipole moment). In Sections 5.5.8, 5.5.9 and 5.5.10, we considered the $\binom{3}{1} = 3$ combinations of moments taken singly (q, d and μ); and then, in Sections 5.5.11, 5.5.12 and 5.5.13, the further $\binom{3}{2} = 3$ combinations of moments taken doubly ($d\mu$, $q\mu$ and qd). It therefore only remains for us to consider the $\binom{3}{3} = 1$ way of taking all three moments triply (*i.e.*, $qd\mu$).

This remaining task is, however, rendered *trivial* by virtue of the fact that the mechanical field density expressions (5.94), (5.95) and (5.96) are all *quadratic* in the fields of the particle: if we insert into them the sums of the fields from the three types of moment, the terms that will result from an expansion via the distributive law will contain the fields of the moments either taken pair by pair, or else quadratically in the field of a single moment; and hence the results for the triply-momented particle are, in fact, the superpositions of corresponding results found in Sections 5.5.8, 5.5.9, 5.5.10, 5.5.11, 5.5.12 and 5.5.13.

We shall, for conciseness, refer to this general propery of quadratic expressions by the term *pairwise superposition*; the mechanical field self-densities are thus *pairwise superposable*.

5.5.15 The relativistic worldline fields

Finally, we consider the problem of obtaining the correct contribution to the retarded field expressions *on the worldline* of the generating point particle. From the analyses of the previous sections, we know that, even for a *static* particle, the dipole moments have a somewhat subtle behaviour at the position of the particle. The question then arises: how are these results modified when the particle is in *arbitrary motion*—*i.e.*, when $\dot{\boldsymbol{v}}$, $\dot{\boldsymbol{\sigma}}$, *etc.*, are not zero?

It is tempting to think that one may proceed simply from the static case, by formulating a Lorentz-covariant expression, and then deeming that this holds in all frames. But this brings us dangerously close to the error that Cohn made in his unfortunate 1969 paper [53]: one *cannot* simply analyse a static system, boost the results, and hope for the best, since the effects of acceleration will then *automatically* be lost—as a static system knows nothing of acceleration, and hence cannot possibly have terms involving it as a factor.

Instead, we must proceed a little more carefully. Our plan of attack is as follows: Firstly, we shall evaluate the full expressions for the arbitrary-motion retarded fields, obtained in Section 5.4.6, for the extended model considered in the previous sections—namely, an infinitesimally small sphere (in its instantaneous rest frame) which has a uniform density of electromagnetic moment. We shall then (for technical reasons) *shrink* these expressions to a point, but then "regularise" them again using a mathematical trick used in standard electrodynamics textbooks. We shall then compute the *three-divergence* of the regularised point particle electric and magnetic fields thus computed: this will, from Maxwell's equations, tell us what source densities are, *assuming the expressions of Section 5.4.6 only*. Finally, we can then *add extra fields*—such as the extra Maxwell field of Section 5.5.3—so that the Maxwell equations indicate the physically correct source current densities.

To perform this procedure, however, we face a problem of a logistical nature, which is somewhat of an embarassment to the author: namely, in carrying out the steps above, we would in fact need to compute almost all of the aspects of the radiation reaction calculations, that is actually going to take up the remaining chapter of this thesis. It may therefore seem that it would have been better if the author had postponed this section to the end of Chapter 6, by which time all of those prerequisite results had in fact been presented. But the problem is that the worldline fields are, themselves, also *inputs* to the radiation reaction calculations. We thus are faced with a veritable Catch 22 situation, in which we must obtain the results of Chapter 6, before that chapter is even begun.

To short-circuit this paradoxical situation, we violate logical causality, and present the findings of the point particle three-divergence computations now, with the promise that the loop will be closed by the end of Section 6.6 of Chapter 6.

It is found, in Section 6.6, that, in the instantaneous rest frame of the point particle, the divergences of the full retarded field expressions of Section 5.4.6 *are*, in fact, identical to those for a static electric dipole. Thus, after somewhat of a theoretical anticlimax, we need only find a relativistic generalisation of the magnetic Maxwell field $\boldsymbol{B}_M(\boldsymbol{r})$, equation (5.93) of Section 5.5.3. Clearly, the function

$$F^{\mu}_{M}(x) = \mu \times U \, \delta(\zeta) \tag{5.106}$$

will perform this task.

Our considerations of the retarded fields are now complete.

Chapter 6

Radiation Reaction

According to our fundamental assumptions, each element of volume of an electron experiences a force due to the field produced by the particle itself, and the question now arises whether there will be any resultant force acting on the electron as a whole.
—— H. A. Lorentz (1906) [137]

6.1 Introduction

If a pointlike charged particle is accelerated, and hence emits electromagnetic radiation, who pays for the mechanical energy contained in the radiated fields? Clearly, the particle itself has to pay; there is no one else around to pick up the bill. But if the point charge is paying out hard-earned mechanical energy, it must be at the expense of its *kinetic* energy—the particle having no other negotiable currency in its possession. It therefore effectively has a *"dragging" force* applied to it, the *reaction of its own emitted radiation*.

As such, the logic of this argument is easy to understand. To figure out *what* the radiation reaction force is, we need simply invoke the methods of Sherlock Holmes: whatever mechanical energy the radiation field eventually flees with *must* be the kinetic energy lost by the particle.

But this situation is not entirely satisfactory. Firstly, if we concentrate our attention on only the *radiation* fields (*i.e.*, those that escape to infinity, that fall off as R^{-1}), then we are ignoring the mechanical energy contained in the *near fields* (those that fall off as R^{-2})—and, indeed, the interference between the near and far fields. Of course, if the system is accelerated for a certain *finite* period of time, and then left alone forever after, the near fields will eventually stabilise themselves, and only the mechanical energy pilfered by the radiation fields will come off the bottom line of the balance sheet for the mechanical energy. But this small amount of information is rarely acceptable to us: we would generally like to know what is happening *now* to the electric charge—as it is radiating—not what it will be doing next week. Thus, we clearly need to expand our considerations, to also take into account the near fields of the charge.

But now we hit another problem: as we approach the position of the point charge, we find that the field (and hence mechanical energy density) expressions *grow divergently*. We must therefore be extremely careful how we evaluate integrals of the mechanical field energy density: we need to "regularise" them in such a way that we avoid any danger of manipulating infinite expressions meaninglessly. One way we can do this is to let the point particle have a *finite size* to start off with, and then shrink it to a point at the end of the calculations. Another way is to *exclude a small region* surrounding the charge, and take into account the energy and momentum crossing the boundary of this region when computing our balance sheets. Whichever way we proceed, we find that we can, in fact, obtain meaningful answers.

But there is another, more deeply philosophical question that one may ask: *who is responsible for this radiation reaction force?* We have figured out what its *effects* are—by some nifty detective work,—but we have not really pinpointed its fundamental *source*. The Lorentz force law—which we can derive, most elegantly, from the interaction Lagrangian—does not give us any clue whatsoever that such a force is lurking on our doorsteps. Where else, then, could it come from? Maybe we have to add *extra* terms to our fundamental Lagrangian, by hand, that yield the correct force? Unfortunately—or, perhaps, fortunately,—such a "fudge factor" approach cannot even be made to work. We are thus left with the Lagrangian on the one hand, describing all of the evolution of the system *except* for the effects of radiation reaction; and the radiation reaction terms on the other, sticking out like the canine's proverbials.

And then Lorentz makes the statement quoted at the head of this chapter. When first heard, it invariably sounds absurd. Imagine, for example, a small sphere, uniformly charged throughout its volume. The Coulomb field of each infinitesimal volume of charge imparts an electric force on every other infinitesimal volume of charge in the sphere. Indeed, the like charges all repel, and the sphere explodes. This is not our intention; let us, therefore, add other, non-electromagnetic forces that balance the Coulomb forces, and hence bind the infinitesimal constituent charges together. These rigid body forces are completely internal, so they do not contribute to the motion of the centre of mass of the body. Lorentz's statement then amounts to saying: let's add up all the Coulomb forces, and see what the resultant net force on the body as a whole is. *But of course it is zero:* the force of the Coulomb field of constituent charge A on constituent charge B is, quite trivially, equal and opposite to that of constituent B on constituent A.

But still Lorentz insists that we should add up the interactions between the constituents of the charge. To extract any physics from such an intuitively useless pastime is the mark of sheer brilliance. One is therefore completely humbled when one realises that one has, in fact, forgotten Maxwell's words, quoted at the head of Chapter 5: *"... without assuming the existence of forces capable of acting directly at sensible distances."* The Coulomb force *is not* instantaneously propagated from sender to receiver; it propagates at the speed of light. The Coulomb force felt *now* by constituent A is determined by the motion of constituent B *as it was some time ago*; the Coulomb force felt now by constituent B is likewise determined by the motion of constituent A as it was some time ago. We may still fail to see, through this observation, any solution to our problem: if A and B are rigidly connected, then surely their respective motions, "some time ago", were identical?

The crucial realisation is that the retarded time of constituent A, as seen by constituent B, is *not*, in general, the same as the retarded time of constituent B, as seen by constituent A, *if the body is being accelerated*. This fact can be recognised from first principles: Imagine that the spherical body, of radius ε, is being accelerated in the positive-x direction, which we shall describe as being "to the right" of the origin; let us take the body to be stationary, centred on the origin, at $t = 0$, which we shall describe as "now"; and let us take, for simplicity, the points A and B at diametrically opposite ends of the sphere, along the x-axis, with A at the position $x = +\varepsilon$, and B at the position $x = -\varepsilon$. Clearly, in the past, the centre of the sphere was always *to the right* of the origin (since, for nonrelativistic motion, we have $x_{\text{centre}} = \frac{1}{2}at^2$; the sphere has reached the minimum of its trajectory, along the x-axis); thus, the points A and B were, in the past, *to the right* of where they are now. But this means that, for constituent B to send a light beam to constituent A, the beam only has to travel a *shorter* distance than their separation (now) of 2ε; and for constituent A to send a light beam to constituent B, the beam has to travel a *longer* distance than 2ε. But the *retarded time*, in naturalised units, is simply equal to the distance the light has to travel; hence, A "sees" B as he was a short time ago, but B "sees" A as he was a longer time ago. This seemingly counterintuitive result is, of course, simply another consequence of Einstein's second postulate: that the speed of light is an invariant, independent of the velocity of the sender.

Now, since the magnitude of the electric field generated by a charge is determined by the velocity and acceleration of the charge as "seen" at the *retarded* time, as well as directly depending (as $1/R$ or $1/R^2$) on this retarded distance itself,—and since the velocity and acceleration are *themselves* changing with time,—then it is clear that the retarded Coulomb force of A on B will, in the general case, be *different* to that of B on A.

Lorentz performed this calculation quantitatively,—using, however, *Galilean* rigidity,—and found, in the rest frame of the body,

$$\boldsymbol{F}_{\text{self}} = -\frac{4}{3}\, m^q_{\text{e.m.}}\, \dot{\boldsymbol{v}} + \frac{2}{3}\, \frac{q^2}{4\pi}\, \ddot{\boldsymbol{v}} + \mathrm{O}(\varepsilon), \tag{6.1}$$

where $m^q_{\text{e.m.}}$ is the static field mechanical self-energy (5.97) of Section 5.5.8, and where the terms of order ε and higher in (6.1) disappear in the point limit $\varepsilon \to 0$.

The properties of (6.1) are, arguably, quite astounding. Firstly, we note that (6.1) has been derived *directly from the Lorentz force law*; thus, while it is has not been obtained *explicitly* from a Lagrangian, it is nevertheless *implicitly* contained in the Lagrangian derivation of the Lorentz force law itself, *when the fields* $\boldsymbol{E}$ *and* $\boldsymbol{B}$ *appearing therein are taken to be the total fields, including the self-fields.*

Secondly, it was shown by Abraham [3] that the second term in (6.1) *completely describes* the force of radiation reaction. (For example, for circular motion, $\dot{\boldsymbol{v}}$ is radially inwards, and hence $\ddot{\boldsymbol{v}}$ is antiparallel to the velocity $\boldsymbol{v}$; the coëfficient $(2/3)(q^2/4\pi)$ gives the correct balancing of the loss of energy.)

Thirdly—and, perhaps, most remarkably—the first term of (6.1) provides a completely *dynamical* explanation of how the "inertial mass" of the charge's self-field works: it provides a real, quantifiable force *opposite to the acceleration* of the charge; one does not have to insert this inertial property by hand. Of course, the factor of 4/3 in this term of Lorentz's result is incorrect; the coëfficient should be unity. It has been shown by a number of authors, in numerous instructive ways, that this is due to an inappropriate use of nonrelativistic concepts; we shall discuss this in greater detail shortly.

It may seem that the author is devoting an unjustifiable amount of effort to simply re-tell a story that is already told in any good electrodynamical textbook (*see*, *e.g.*, [113]). Perhaps so. But the author wishes to emphasise most strongly that the Lorentz method of derivation of the radiation reaction force is not only *more powerful* than the method of considering the mechanical stress-energy tensor of the field (perfected, in the fully relativistic case, by Dirac [68])—the latter requiring reasonable but underivable assumptions to be made, and arbitrary constants to be fitted; the Lorentz method is also *intuitively simple to understand*. The relativistic shortcomings of Lorentz's original derivation are simply repaired, using the formalism of Chapter 3; one is then left with a complete, rigorous, intuitive and æsthetically pleasing method of derivation of the radiation reaction equations of motion.

It is this method that we shall use, in this final chapter, to derive the classical radiation reaction equations of motion for point particles carrying electric charge and electric and magnetic dipole moments. In Section 6.2, we review the work of Bhabha and Corben, who considered this problem in 1941, using the Dirac stress-energy method; and a related analysis undertaken recently by Barut and Unal, using not the classical spin formalism of this thesis, but rather a semiclassical "*zitterbewegung*" model of spin. We then attack the problem anew, beginning, in Section 6.3, with a consideration of various aspects of the use of an infinitesimal rigid body. In Section 6.4 we introduce the sum and difference constituent position variables, and show that their use is not a trivial as one might naïvely expect. We then compute, in Section 6.5, the retarded kinematical quantities of the constituents of the body, and use these results to obtain the retarded self-field expressions. The three-divergences of these expressions are computed in Section 6.6, as the final step of the computation of the relativistic worldline fields considered in Chapter 5. Some necessary subtleties involved with the integration of terms in the analysis of an inverse-*cube* dependence are discussed in Section 6.7. In Section 6.8 we compute the radiation reaction equations of motion themselves; these are discussed in Section 6.9. Finally, in Section 6.10, we apply one of the equations obtained to the Sokolov–Ternov and Ternov–Bagrov–Khapaev effects, and highlight both the successes and limitations of the completely classical analysis.

6.2 Previous analyses

To the author's knowledge, the classical radiation reaction problem for particles carrying dipole moments has only been considered twice before: by Bhabha and Corben [40] in 1941, using the Dirac stress-energy method of derivation; and in a related analysis in 1989 by Barut and Unal [35], which did not consider the problem on completely classical grounds, but instead employed a semiclassical "*zitterbewegung*" model of spin.

In Section 6.2.1, we briefly review the work of Bhabha and Corben, following by a review of the Barut–Unal analysis in Section 6.2.2.

6.2.1 The Bhabha–Corben analysis

As should by now be apparent, Bhabha and Corben attacked most of the oustanding problems of classical particles carrying dipole moments in their 1941 paper [40]. The question of the radiation reaction is no exception.

Bhabha and Corben used the method developed by Dirac only a few years earlier [68], of surrounding the worldline of the particle with a small tube, and considering the mechanical energy, momentum and angular momentum crossing this tube, for arbitrary motion of the particle. From conservation requirements, one can then deduce the force and torque on the particle.

The method of Dirac was, at the time, a vast improvement on the Lorentz method of derivation, as it was *manifestly covariant*. The resulting covariant radiation reaction equation of motion for a charged particle is for this reason referred to as the *Lorentz–Dirac equation*. However, the basic Dirac method also has its drawbacks, alluded to in Section 6.1. In its raw form, it requires the form and coëfficient of the *inertial* term to be simply guessed at (*see* Dirac's derivation [68]). This problem of arbitrariness has been overcome, in more recent times, by considering the retarded and advanced fields with somewhat more care [203, 204, 205]; or, somewhat more elegantly, mathematically speaking, by performing an analytical continuation of the fields right up to the worldline of the particle [28, 29, 30]. Of course, these more advanced techniques were not available to Bhabha and Corben.

The Bhabha–Corben analysis of the radiation reaction question is, as with all parts of their paper, performed completely in manifestly covariant form. The argumentation is very clear and straightforward to understand, as are some of the final results that they analysed for special circumstances (*e.g.*, a free particle), but their general results may only be described as "lengthy". Indeed, they relegate even the *listing* of the results to the Appendix; the following comment of theirs summarises the flavour of the algebra involved [40]:

> Since $F^{(2)}_{\mu\nu}$ given by (114) contains no less than 18 terms, $T_{\mu\nu}$ which is quadratic in $F_{\mu\nu}$ contains some 324 terms. Some of these, of course, vanish at once from symmetry, or due to (18), but nevertheless the calculation is very lengthy and tedious. We have not found a way of shortening it.

The author will, indeed, have similar comments to make by the end of this chapter; fortunately, the *final* expressions of the author will not be quite as complicated as this.

The explicit results listed by Bhabha and Corben are not transparently understandable—except for the electric charge results, of course, which yield simply the Lorentz–Dirac equation. The Bhabha–Corben covariant force and torque expressions are actually given not as final expressions, as with the Lorentz–Dirac equation, but rather as expressions containing many *unperformed proper-time derivatives* of terms, each containing a number of kinematical factors. The said force and torque expressions each contain 91 terms, which between them contain 417 kinematical factors which would require proper-time differentiation by the product rule in order for one to obtain explicit results. It it therefore not too surprising that the Bhabha–Corben results do not appear to have been put to any substantial practical purpose, other than pedagogy, in the past fifty-three years.

The Bhabha–Corben analysis, however, appears to most definitely be based on a correct method of derivation. Whilst one cannot know whether their algbraic manipulations of such lengthy expressions can be fully relied on, the correct physics *should* be in there—somewhere. We therefore examine only the broader features of their results. Firstly, they find that time-derivatives up to the *fourth* order are required in the force equation, and up the *third* order in the torque equation. Secondly, the *coëfficients* of the terms that they find all have denominators from the following set of numbers:

$$1, 3, 5, 6, 15, 35.$$

Thirdly, they find numerous terms that depend on ε^{-3} and ε^{-1}, but only a few dependent on ε^{-2}. They state that these terms were "expected"; the author is still coming to grips with the detailed arguments for them.

Beyond this, it is difficult to go, without examining the Bhabha–Corben equations in explicit detail.

It is clear that any new classical analysis of this problem should be compared against the Bhabha–Corben results. However, due to the complexity of their findings, such a comparison is not performed in this thesis, other than a comparison of the broad features listed above. It is, perhaps, left as an exercise for the reader!

6.2.2 The Barut–Unal analysis

In 1989, Barut and Unal [35] considered anew the question of the generalisation of the Lorentz–Dirac equation to particles possessing spin.

Their method of doing so, however, did not follow the completely classical path of Bhabha and Corben (which is also to be followed by the author in this chapter). Rather, Barut and Unal considered a "*zitterbewegung*" model of spin, developed by Barut and collaborators in previous years [31, 33, 34, 32, 36]. This model of spin incorporates the simplicity of classical trajectories of *translational* motion, while incorporating the *spin* degrees of freedom *à la* the Dirac equation, by means of an elegant symplectic formalism, *not* by using the classical spin vector or tensor.

The radiation reaction equation of motion found by Barut and Unal is

$$\dot{\pi}^{\mu} = eF^{\mu\nu}_{\text{ext}} v_{\nu} + \alpha \tilde{g}_{\mu\nu} \left\{ \frac{2}{3} \frac{\ddot{v}_{\nu}}{v^2} - \frac{9}{4} \frac{(v \cdot \dot{v}) \dot{v}_{\nu}}{v^4} \right\}. \tag{6.2}$$

where $\alpha \equiv e^2/4\pi$, and where

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} - \frac{v_{\mu} v_{\nu}}{v^2}.$$

We have not converted the notation of the equation (6.2) to the conventions of this thesis, because the quantities involves are not of quite the same nature. The "four-velocity" v_{μ} appearing in (6.2) exhibits the "*zitterbewegung*" motion of the model, in a similar way that the velocity operator of the Dirac equation does in the Dirac–Pauli representation. The Barut–Unal four-velocity does *not* satisfy $v^2 = 1$; indeed, it is not even real: its complex oscillation represents the spin angular momentum of the particle. It is for this reason that Barut and Unal only require *one* equation of motion, namely, (6.2): it contains the reaction on the translational *and* the spin motion in one equation.

The author must now pass comment on the Barut–Unal analysis. Firstly, the author cannot see much *practical use* in the Barut–Unal equation (6.2), as it stands. The equation is a remarkably simple expression of the effects of radiation reaction, but the "four-velocity" involved has no direct connection with the four-velocity measured by experimentalists in the real world. From the discussion of Section 4.4.1, it should be clear that the author considers the concept of "*zitterbewegung*", as a classical explanation, dubious at the best of times. Now, if the Barut–Unal equation were to be subjected to the semiclassical equivalent, in the Barut model, of some sort of a "Foldy–Wouthuysen transformation", then the resulting equations of motion *would* be of immediate practicality, since then we would again have position, velocity, mechanical momentum and spin quantities that would have a direct connection with classical physics. But the author has not seen such a suitable transformation provided.

The author's ire is raised beyond the point of containment by several comments in the Barut–Unal paper [35]. Let us reproduce them here for the consideration of the reader:

> In the Bhabha–Corben equations, v^2 is taken to be unity; hence $v\cdot\dot{v} = 0$ and the term with coefficient $\frac{9}{4}$ is missing, and $v_\mu S^{\mu\nu} = 0$. Both of these relations do not hold here; the BC equation is an approximation to ours.

And the concluding paragraph, in full:

> Our main result is Eq. (12) [or (18)]. We believe that it is the first relativistic symplectic formulation of both coordinates and spin and the first significant generalization of the Lorentz–Dirac equation since 1938. In the second term of (12) we have the LD term $\frac{2}{3}(\ddot{v}^\nu/v^2)\tilde{g}^{\mu\nu}$ but also the *new term* $-\frac{9}{4}[(v\cdot\dot{v})\dot{v}^\nu/v^4]$. Another difference from the LD equation is that on the left-hand side we have $\dot{\pi}^\mu$ instead of $m\ddot{x}^\mu$. The Bhabha–Corben equation is not derived from an action principle, but from considerations of energy conservation of a magnetic dipole moment, and the new term $-\frac{9}{4}[(v\cdot\dot{v})\dot{v}^\nu/v^4]$ is missing. They have assumed a mass point with charge g_1 and dipole moment g and put $v^2 = 1$, $S_{\mu\nu}S^{\mu\nu} = 0$, and $S_{\mu\nu}v^\nu = 0$ from the beginning.

What rot. *Of course* Bhabha and Corben take $v^2 = 1$: they are considering standard classical mechanics. To describe the term with coëfficient $9/4$ as "missing" carries with it the completely misleading implication that Bhabha and Corben obtained simply the Barut–Unal result, but with that single term omitted! This attempt at self-aggrandisement fails as soon as the reader consults the original Bhabha and Corben paper, to find out what they *actually* found. The only similarity between the Bhabha–Corben results, and the Barut–Unal equation (6.2), is that the *Lorentz–Dirac* equation for a charge is contained in the Barut–Unal result when $v^2 = 1$. The Bhabha–Corben results for *dipole* moments contain—as described and lamented in the previous section—*dozens* of terms, over and above the Lorentz–Dirac equation. These are conveniently ignored by Barut and Unal.

As to the statement that "the BC equation is an approximation to ours", one wonders whether Barut and Unal actually read the Bhabha–Corben paper at all. The Barut–Unal and Bhabha–Corben results appear to *both* contain the correct physics, but in completely different ways. To state that one is more accurate than the other would require them to be expressed in similar form, and the conclusions compared. The author has already suggested above that a semiclassical "Foldy–Wouthuysen transformation" of the Barut "*zitterbewegung*" model is desirable; if performed, a direct comparison on classical terms could be made. It would be difficult to see how one could transform the Bhabha–Corben results to the Barut model. In any case, a direct comparison between the Bhabha–Corben and Barut–Unal results cannot be made at this stage.

The author agrees that the Barut–Unal analysis may well be "the first relativistic symplectic formulation of both coordinates and spin", for this application. But to state that it is "the first significant generalization of the Lorentz–Dirac equation since 1938" is yet another self-aggrandising attempt to rewrite history.

In summary, the author considers the mathematical techniques of Barut and Unal to be elegant and ingenious, and a promising start to a new line of attack on this problem. But he finds their comments reprehensible.

6.3 Infinitesimal rigid bodies

We now consider afresh the problem of deriving the radiation reaction equations of motion for a point particle carrying electric charge and electric and magnetic dipole moments, based on the results of the previous chapters.

From the discussion of Section 6.1, we know that it is necessary to "regularise" a pointlike particle, before we compute self-fields or self-interactions, if one is to obtain meaningful answers. Since we have chosen to follow the Lorentz self-interaction method of derivation of radiation reaction, it is only natural that we employ his technique of *expanding the point particle into a small rigid sphere*, of radius ε.

Now, in Chapter 3, we considered the various problems associated with defining rigidity in a relativistically meaningful way. It was found that, indeed, it *is* possible to define rigidity in a meaningful way, but that the constituents of such a rigid system may well end up "crossing the accelerative horizon" if the body is subject to a sufficiently large acceleration.

If we consider an *infinitesimal* relativistically rigid body, however, such a problem disappears, in the transition to the point limit. For the acceleration $\dot{\boldsymbol{v}}$ of the system is a *finite*, "external" quantity, that *does not* vary as we take the point limit; hence, the "accelerative horizon" is, at any time of a particle's motion, at some particular fixed distance from the centre of the system; and thus, as we shrink the body smaller and smaller, all of its constituents must eventually, at some point of this transition, be contained completely within this horizon.

6.3.1 Orders of expansions

If we are solely concerned with interactions, propagated at the speed of light, *within* the infinitesimal sphere—as we are for the self-interaction calculations,—then clearly the *maximum* temporal period that will be of relevance to us will be on the order of the time required for light to cross the stationary body—which, in naturalised units, is just 2ε. (From the arguments of Section 6.1, it is clear that times *greater* than this static light-distance are relevant, but they are all larger by a dimensionless factor: they are still of order ε.) The *spatial distances* of relevance to us are also, clearly, of order ε.

Thus, in distinction to the analysis of Chapter 3, where we merely expressed the system's trajectory in terms of a Taylor series in time for convenience, for the current application we may expand our expressions out *both temporally and spatially* in the small parameter ε—and, more importantly, we can extract *exact* self-interaction results, in the point limit $\varepsilon \to 0$, since any terms of order ε or higher in these final expressions will vanish rigorously in this limit.

6.3.2 Electromagnetic moment densities

There arises the question, in any use of a spherical body in the regularisation of a point particle, of how one is to distribute the electromagnetic moments throughout the interior of the volume. The *external fields* are, for any sensible choice of distribution, unconcerned about how the sources are arranged (for the case of the electric charge, this is most simply recognised by considering Gauss's law); but the computed mechanical self-field quantities *are* of course modified, since they have contributions from the fields both internal and external to the sphere, which are the same order of magnitude (*see*, *e.g.*, Section 5.5 of Chapter 5).

For the case of simply an electric charge, a *spherical charge shell* is often employed (*i.e.*, a sheet of charge around the surface of the sphere). With such a choice, the charge density is still infinite on the surface, but this infinity is now one-dimensional, rather than three-dimensional as with a point charge. This reduction in pathology *is*, in fact, sufficient for one to analyse the system without infinities entering the expressions of relevance. It also has the added benefit that, by Gauss's law, there is *no electric field inside the sphere*, and hence one does not have any "internal" contribution to the mechanical self-energy.

For the purposes of this thesis, however, a spherical shell of source density is less than desirable. Firstly, we do not know, *a priori*, that the infiniteness of the source density on the surface is sufficiently benign for the physical expressions arising from the presence of *dipole moments* to be sufficiently regularised. Secondly, we *already* have numerous elementary analyses of of the static self-fields and mechanical self-quantities of dipole moments *uniformly* distributed thoughout the spherical volume (*see*, *e.g.*, [113], or the analysis of Section 5.5). Thirdly, and ultimately most relevantly, the author cannot quite understand why one would wish to use a shell density instead of a uniform density anyway: all one could possibly achieve with such a choice is an introduction of bothersome delta-function contributions to one's expressions on the boundary surface of the volume of integration, the pitfalls of which one could no doubt avoid by clever footwork, but which one can avoid even more simply by not digging them for oneself in the first place. In any case, it will be found that we shall *already* find a sufficient number of subtleties involving infinities to deal with, without introducing more of them.

Thus, we shall, for the remainder of this thesis, consider the electric charge, electric dipole moment and magnetic dipole moment to be *uniformly distributed* throughout the spherical volume:

$$a_\rho(\boldsymbol{r}) = \begin{cases} \dfrac{3a}{4\pi\varepsilon^3}, & r < \varepsilon, \\ \\ 0, & r > \varepsilon, \end{cases} \tag{6.3}$$

where $a = q, d, \mu$ is the electric charge, electric dipole moment or magnetic dipole moment of the particle respectively, and we have noted that the volume of a sphere of radius ε is $4\pi\varepsilon^3/3$.

6.3.3 Sending and receiving constituents

It follows from the discussion of Section 6.1 that a calculation of the self-interaction equations of motion involves two steps: Firstly, we need we need to consider the *fields generated* by all of the constituents, as given by the retarded field expressions of Chapter 5. Secondly, we need to compute the *forces* that these fields exert on all of the other constituents of the body. (We shall consider the term "forces" to encompass the three related concepts of "force on", "power into" and "torque on" a particle, where unambiguous; there does not appear to be another word covering this concept with any less ambiguity).

To link the two parts of our calculation, we must essentially determine the path travelled by the retarded fields in going from "sender" to "receiver" (as we shall henceforth call them); from this information, we can compute the appropriate kinematical quantities of the sending and receiving constituents, which appear in the generated field and received force expressions respectively.

From the considerations of Chapter 3, it is clear that the *forces* on the body are most simply considered at the instant that the body is at rest with respect to the Lorentz frame that one chooses. We shall therefore choose precisely such a frame in which to base our considerations. The centre of the sphere is, as in Chapter 3, considered to be at the origin of spatial and temporal coördinates, and the receiving constituents are labelled by their three-position $\boldsymbol{r}$ in this rest frame:

$$|\boldsymbol{r}| \leq \varepsilon.$$

Now, consider a particular constituent $\boldsymbol{r}$. This constituent will "see" all of the other constituents of the body, at their various retarded times: a short time ago for those constituents near to $\boldsymbol{r}$; a longer time ago for those constituents far from $\boldsymbol{r}$; and, from the discussion of Section 6.1, the particular times depend on the acceleration, *etc.*, of the body as a whole. Now, the forces on the receiving constituent $\boldsymbol{r}$, due to the retarded fields generated by all of the other constituents, are to be summed together to obtain the net forces on the constituent $\boldsymbol{r}$; and then, in turn, the forces on all of the constituents $\boldsymbol{r}$ are to be weighted by the accelerative redshift factor $\lambda(\boldsymbol{r})$, and then summed together to obtain the forces on the body as a whole.

Clearly, in carrying out this procedure, we must have some way of labelling the *generating* constituents of the body. We are not talking about the task of finding the actual retarded time, retarded position, *etc.*,—this will be considered in the following sections; rather, we simply need some way to *identify* the generating constituents. This may seem trivial, but the situation is somewhat subtle. To perform the first-mentioned sum above—namely, that of all of the forces on some particular constituent $\boldsymbol{r}$ due to the retarded fields of all of the other constituents—we need to perform some sort of integral over the "sending" sphere. But the various constituents of this "sending" sphere are all "seen" as they were at *different* times in the past; we must effectively integrate the "sending" sphere over a quite complicated spacetime hypersurface, *not* over one of its rest-hypersurfaces. And then we must ensure that we have correctly calculated the relevant transformation properties of the *source densities*, over this complicated hypersurface.

The way out of this conceptual nightmare is to *label* the sending constituents by the simple three-vector $\boldsymbol{r}'$, which represents *the three-position of the sending constituent in the rest frame of the body*:

$$|\boldsymbol{r}'| \leq \varepsilon.$$

One then computes the particular retarded four-position of this sending constituent in order that one can compute the generated fields; but one does *not* try to use this four-position as the variable for integrating over. Rather, the "sending" sphere integral is simply one over $\boldsymbol{r}'$:

$$\int_{r'\le\varepsilon} d^3r'.$$

This procedure automatically ensures that the source densities are correct (in fact, they are then trivially constant, over this volume), while still yielding a relativistically correct evaluation of the self-interactions—which will thus be of the form

$$\frac{ab}{4\pi}\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\le\varepsilon} d^3r\int_{r'\le\varepsilon} d^3r'\,\lambda(\boldsymbol{r})f(\boldsymbol{r},\boldsymbol{r}'), \tag{6.4}$$

where $a, b = q, d, \mu$ (a being the moment generating the fields, b the moment receiving them), and $f(\boldsymbol{r},\boldsymbol{r}')$ is the computed force (or power, torque) on the constituent $\boldsymbol{r}$ due to the retarded fields of the constituent $\boldsymbol{r}'$. (We include the ever-inconvenient factor of $1/4\pi$ in (6.4), as it always appears due to the similar factor present in the retarded field expressions. As with the expressions of Chapter 5, we shall absorb this factor into our notational definitions shortly.)

The difference between the *label* three-vector $\boldsymbol{r}'$, and the relative three-spatial position of this constituent with respect to the centre of the body $\Delta\boldsymbol{z}_{r'}(t_{\text{ret}})$, of course trivially *vanishes* for the Galilean model of rigidity used by Lorentz in his original self-interaction calculations. It is thus actually due the requirements of *relativistic invariance* that our considerations of this section are more subtle than those of Lorentz [137].

We shall not comment here on the irony of Lorentz using the Galilean rather than the Lorentz transformation in his computations.

6.4 The sum and difference variables

Before we launch into the detailed algebraic calculations of the retarded self-fields and consequent self-interactions, for arbitrarily complicated motion of the particle, it is instructive to first consider a much simpler case, to ground our understanding of the quantities involved: the case of a *stationary* particle. Now, from the discussion of Section 6.1, we know that this example is trivial: there is no self-interaction at all. But the fundamental quantities involved in obtaining this null result are, naturally, the zeroth-order terms of the corresponding quantities when arbitrary motion *is* considered, and hence will provide us with a leading-order intuitive understanding of the complicated algebraic expressions to be considered shortly.

Let us first consider the retarded field experienced by the receiving constituent $\boldsymbol{r}$ due to the retarded sending constituent $\boldsymbol{r}'$. The relative separation three-vector from $\boldsymbol{r}'$ to $\boldsymbol{r}$ is, of course, simply

$$\boldsymbol{r} - \boldsymbol{r}'.$$

This will, in the static case, be the quantity $R\boldsymbol{n}$ of the retarded field expressions, according to the notation of Chapter 5. Let us make the simple notational definition

$$\boldsymbol{R} \equiv R\boldsymbol{n}. \tag{6.5}$$

Then, from the above, we have

$$\boldsymbol{R}(\boldsymbol{r}, \boldsymbol{r}')|_{\text{static}} = \boldsymbol{r} - \boldsymbol{r}'. \tag{6.6}$$

Now, the *time* that it takes for the electromagnetic field to propagate from the sending constituent $\boldsymbol{r}'$ to the receiving constituent $\boldsymbol{r}$ is simply given (in naturalised units) by R, the distance that the light has to travel. We can extract R from (6.6) by taking the magnitude of the vector:

$$R(\boldsymbol{r}, \boldsymbol{r}') \equiv |\boldsymbol{R}(\boldsymbol{r}, \boldsymbol{r}')|\,;$$

in the static case, (6.6) then yields

$$R(\boldsymbol{r}, \boldsymbol{r}')|_{\text{static}} = |\boldsymbol{r} - \boldsymbol{r}'| \equiv \left\{(\boldsymbol{r} - \boldsymbol{r}')^2\right\}^{1/2}. \tag{6.7}$$

Since we have set our origin so that the *receiving* constituents are all at $t = 0$, the *sending* constituents are "seen", by the receiving constituents, at the (earlier) retarded times $t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}') < 0$. In particular, we have

$$t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}') \equiv -R(\boldsymbol{r}, \boldsymbol{r}'), \tag{6.8}$$

—not just in the static case, but in general: (6.8) simply says that the speed of light is unity, in naturalised units. From (6.7) and (6.8), we thus have, for the *static* case,

$$t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}')\big|_{\text{static}} \equiv -\left\{(\boldsymbol{r} - \boldsymbol{r}')^2\right\}^{1/2}. \tag{6.9}$$

Now, when we shortly consider the case of *arbitrary* motion, we shall of course be employing the Taylor series expansions of Chapter 3 for the motional and spin trajectories of the various constituents $\boldsymbol{r}'$, as functions of time. To obtain the retarded kinematical quantities of the sending constituent $\boldsymbol{r}'$, as seen by the receiving constituent $\boldsymbol{r}$, we will then simply substitute the retarded time $t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}')$ into these trajectory expressions.

In the static case, the results are trivial: the kinematical properties of the particle are constant throughout time. But just imagine if we *were* to substitute even the *static* retarded time expression (6.9) into some Taylor series expression in t: we would end up with terms looking like

$$\left\{(\boldsymbol{r} - \boldsymbol{r}')^2\right\}^{1/2}, \ (\boldsymbol{r} - \boldsymbol{r}')^2, \ \left\{(\boldsymbol{r} - \boldsymbol{r}')^2\right\}^{3/2}, \ (\boldsymbol{r} - \boldsymbol{r}')^4, \ \ldots, \tag{6.10}$$

for the polynomials t, t^2, t^3, t^4, $\ldots$, respectively.

Now, the terms (6.10) are somewhat ugly and awkward, but they are nevertheless manageable. The problem arises when we wish to *compute the integrals (6.4)*. Would *you* like to have to integrate terms like those in (6.10), over a spherical volume of $\boldsymbol{r}$, and then over another spherical volume of $\boldsymbol{r}'$? They are not simple.

Instead, to simplify the integrals considerably, one generally defines the *sum and difference* constituent three-vector variables:

$$\boldsymbol{r}_d \equiv \boldsymbol{r} - \boldsymbol{r}',$$

$$\boldsymbol{r}_s \equiv \boldsymbol{r} + \boldsymbol{r}', \tag{6.11}$$

whence the reverse transformation is

$$\begin{aligned} \boldsymbol{r} &\equiv \frac{1}{2}\Big(\boldsymbol{r}_s + \boldsymbol{r}_d\Big)\,, \\ \boldsymbol{r}' &\equiv \frac{1}{2}\Big(\boldsymbol{r}_s - \boldsymbol{r}_d\Big)\,. \end{aligned} \tag{6.12}$$

Then the integrals, over $\boldsymbol{r}_d$- and $\boldsymbol{r}_s$-space, of the terms in (6.10) are much simpler: the terms all involve $\boldsymbol{r}_d$, and are independent of $\boldsymbol{r}_s$.

Of course, when we perform the full calculations, we will find that not *all* terms will possess these simplifying qualities—recall, we only substituted the *static* retarded time expression above; nevertheless, most terms *will*, in fact, be of this form, and the remainder can be treated as special cases.

The change of variable (6.11) may seem obvious; indeed, it is usually glossed over in most accounts of the Lorentz method of derivation. However, *it is, in reality, far from trivial*. In the following sections we examine the subtleties accompanying any use of the transformation (6.11), without due regard for which one cannot validly proceed any further whatsoever.

6.4.1 The six-dimensional hypervolume

The problem with the tranformation (6.11) from $\boldsymbol{r}$–$\boldsymbol{r}'$ space to $\boldsymbol{r}_d$–$\boldsymbol{r}_s$ space is that our original integrals over $\boldsymbol{r}$ and $\boldsymbol{r}'$ were each over *finite* volumes—namely, two independent three-spheres—rather than over all space. The regions of integration for $\boldsymbol{r}_s$ and $\boldsymbol{r}_d$ are thus *dependent*, rather than independent.

The essence of the complication can be seen by considering the simplified case in which $\boldsymbol{r}$ and $\boldsymbol{r}'$ are one-dimensional, rather than three-dimensional: the double-integral

$$\int_{-\varepsilon}^{\varepsilon} dx \int_{-\varepsilon}^{\varepsilon} dx'$$

is over a square region in the x–x' plane; the domains of x and x' are each from $-\varepsilon$ to ε, independent of the value of the other. If we define the new variables

$$x_d \equiv x - x',$$
$$x_s \equiv x + x',$$

the domains of x_d and x_s are now clearly from -2ε to 2ε. The Jacobian

$$\frac{\partial(x,x')}{\partial(x_s,x_d)} = \frac{1}{2}$$

balances only *half* of this apparent fourfold increase in the area of integration. The other half is of course due to the fact that the original *square* area of integration has been rotated into a *diamond* shape. The consequence is that the limits of integration of the "inner" integral are no longer constant, but are rather dependent on the value of the variable in the "outer" integral. Explicitly, if we perform the x_d integral outermost, the limits of integration for the inner x_s integral are

$$|x_s| \leq 2\varepsilon - |x_d|\,;$$

the domain of integration for x_s is now, on the average, half as long as the naïve maximum-minus-minimum calculation would suggest.

The results of the equivalent considerations in the fully three-dimensional case are not as obvious—because we are not so adept at visualising six-dimensional geometry,—but nevertheless proceed in the same manner. The self-interactions that we are considering in this chapter are of the general form (6.4), namely

$$\frac{ab}{4\pi}\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r'\,\lambda(\boldsymbol{r})f(\boldsymbol{r},\boldsymbol{r}').$$

We define

$$\eta_0 \equiv \frac{1}{4\pi}\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r'$$
$$\equiv \frac{1}{4\pi}.$$

The six-dimensional region of integration in $\boldsymbol{r}$–$\boldsymbol{r}'$ space is defined by the constraints

$$|\boldsymbol{r}| \leq \varepsilon,$$
$$|\boldsymbol{r}'| \leq \varepsilon. \tag{6.13}$$

The transition to the variables $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$ mixes components of $\boldsymbol{r}$ and $\boldsymbol{r}'$. Since we know, from the discussion of the previous section, that the three-vector $\boldsymbol{r}_d$ will dominate our considerations, with $\boldsymbol{r}_s$ playing a considerably lesser rôle, we shall usually choose to perform the the $\boldsymbol{r}_d$ integral outermost. The region of integration of $\boldsymbol{r}_d$ will then be determined only by the maximum values attainable by $\boldsymbol{r}_d$; but the region of integration for $\boldsymbol{r}_s$ will depend on the *value of* $\boldsymbol{r}_d$ *in the outer integral*. It is clear, from the definitions (6.11), that $\boldsymbol{r}_d$ may take any value within a three-sphere of radius 2ε, *i.e.*,

$$r_d \leq 2\varepsilon, \tag{6.14}$$

and thus this larger three-sphere is the region of integration for $\boldsymbol{r}_d$; we shall call this volume V_d. The constraints (6.13) require that

$$|\boldsymbol{r}_s + \boldsymbol{r}_d| \leq 2\varepsilon, \tag{6.15}$$
$$|\boldsymbol{r}_s - \boldsymbol{r}_d| \leq 2\varepsilon; \tag{6.16}$$

i.e., the region of integration is the intersection of these two regions. Now, for any given value of $\boldsymbol{r}_d$, these constraints define two three-spheres in $\boldsymbol{r}_s$-space, each of radius 2ε, that are *offset* by $\pm\boldsymbol{r}_d$ from the origin; their common volume—which looks like a three-sphere with a central slice taken out and the remaining pieces glued together—is the volume of integration for $\boldsymbol{r}_s$; we shall call this intersection volume V_s. Taking into account the Jacobian (of value $1/2^3 = 1/8$), we thus have

$$\int_{r\le\varepsilon} d^3r \int_{r'\le\varepsilon} d^3r' = \frac{1}{8}\int_{V_d} d^3r_d \int_{V_s} d^3r_s\,. \tag{6.17}$$

6.4.2 Symmetry arguments

Clearly, integrals in $\boldsymbol{r}_d$–$\boldsymbol{r}_s$ space are not straightforward with such a strange region of integration as that described in the previous section: at the least, we have lost spherical symmetry for $\boldsymbol{r}_s$.

However, there is one consideration that allows us to simplify matters considerably. It will be noted that, since the two defining three-spheres in $\boldsymbol{r}_s$-space are offset by $\pm\boldsymbol{r}_d$, then a parity operation $\boldsymbol{r}_d \to -\boldsymbol{r}_d$ in $\boldsymbol{r}_d$-space leaves the $\boldsymbol{r}_s$ integration region (and hence the value of the inner $\boldsymbol{r}_s$ integral) *invariant*. Since the $\boldsymbol{r}_d$ integral is itself over a three-sphere, we thus find that *integrals odd in* $\boldsymbol{r}_d$ *vanish*, regardless of their dependence on $\boldsymbol{r}_s$.

Likewise, we could have alternatively presented the above with the $\boldsymbol{r}_s$ integral outermost (since, fundamentally, the definitions (6.11) of $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$ are symmetrical), and thus conclude that *integrals odd in* $\boldsymbol{r}_s$ *vanish*, again regardless of their dependence on $\boldsymbol{r}_d$.

6.4.3 Explicit inner integrals

We now consider the explicit expressions involved in performing integrals over the hypervolume specified by V_d and V_s.

We consider the $\boldsymbol{r}_d$ integral to be performed "outermost", and the $\boldsymbol{r}_s$ integral "innermost". The process may be visualised by analogy with a computer program with six nested loops: we are here specifying that the three components of $\boldsymbol{r}_d$ are to be incremented in the three outermost loops; the three components of $\boldsymbol{r}_s$ are incremented in the three innermost loops. The starting and ending values of each loop are not fixed, but rather are determined by the values of the counters in all of the loops that are *outside* the loop in question. Inside the innermost loop is the function itself, which is evaluated for each value of the counters in the six loops, and the resulting value added to the running sum.

We shall, to simplify the following considerations, construct a *new set of spatial axes* (x, y, z) for $\boldsymbol{r}_s$-space, in such a way that the particular values that the components of $\boldsymbol{r}_d$ have, as we enter the $\boldsymbol{r}_s$ integration, are such that the vector $\boldsymbol{r}_d$ is in the z-direction; we parametrise $\boldsymbol{r}_d$ in terms of the new dimensionless quantity α:

$$\boldsymbol{r}_d = 2\varepsilon(0, 0, \alpha);$$

i.e., $0 \leq \alpha \leq 1$. The particular set of axes (x, y, z) used for one "entry" to the innermost three loops is not, of course, the same as that for another "entry"; but they are always related by a rotation. Clearly,

$$\alpha \equiv \frac{r_d}{2\varepsilon} \tag{6.18}$$

in general.

In terms of this new set of axes, we perform the z-integral outermost, followed by the y-integral, and then finally the x-integral. Now, the volume V_s specified by (6.15) and (6.16) clearly has its extremities on the z-axis at values $\pm 2\varepsilon(1 - \alpha)$. For a given value of z, the region of integration in the x–y plane will then be the interior of a circle, of radius ρ, where

$$\rho^2(z, \alpha) = 4\varepsilon^2 - (|z| + 2\varepsilon\alpha)^2.$$

We thus find that

$$\int d^3r \int d^3r' = \frac{1}{8} \int_{V_d} d^3r_d \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \int_{-\rho(z,\alpha)}^{\rho(z,\alpha)} dy \int_{-\sqrt{\rho^2(z,\alpha)-y^2}}^{\sqrt{\rho^2(z,\alpha)-y^2}} dx, \tag{6.19}$$

where the final two sets of integration limits simply specify the interior of a circle of radius $\rho(z, \alpha)$.

We shall, at this point, perform a small feat of mathematical clairvoyance, and predict that the *only* even integrals that we shall ultimately have need to perform in $\boldsymbol{r}_s$-space, in the remainder of this thesis, are of two types: those independent of $\boldsymbol{r}_s$, namely,

$$\int_{V_s} d^3 r_s \, 1; \tag{6.20}$$

and those with two factors of $\boldsymbol{r}_s$, namely,

$$\int_{V_s} d^3 r_s \; r_s^i r_s^j, \tag{6.21}$$

with *no* extra factors of the magnitude r_s in either case. For the case (6.20), we find

$$\begin{aligned}
\int_{V_s} d^3 r_s \, 1 &= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \int_{-\rho(z,\alpha)}^{\rho(z,\alpha)} dy \int_{-\sqrt{\rho^2(z,\alpha)-y^2}}^{\sqrt{\rho^2(z,\alpha)-y^2}} dx \\
&= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \int_{-\rho(z,\alpha)}^{\rho(z,\alpha)} dy \;\; 2\sqrt{\rho^2(z,\alpha) - y^2} \\
&= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \quad \pi\Big\{4\varepsilon^2 - (|z| + 2\varepsilon\alpha)^2\Big\} \\
&= \frac{4}{3}\pi(2\varepsilon)^3 \left\{1 - \frac{3}{2}\alpha + \frac{1}{2}\alpha^3\right\};
\end{aligned}$$

hence,

$$\int_{V_s} d^3 r_s \, 1 = \frac{4}{3}\pi(2\varepsilon)^3 \left\{1 - \frac{3}{2}\left(\frac{r_d}{2\varepsilon}\right) + \frac{1}{2}\left(\frac{r_d}{2\varepsilon}\right)^3\right\}, \tag{6.22}$$

where we have used (6.18) to replace α by $r_d/2\varepsilon$. Although unfamiliar, the result (6.22) can be verified to have intuitively correct properties. For $r_d = 0$, the region of integration in $\boldsymbol{r}_s$-space is just a three-sphere of radius 2ε, and (6.22) gives the expected volume $4\pi(2\varepsilon)^3/3$. For $r_d = 2\varepsilon$, the two offset three-spheres cease to intersect, and the volume (6.22) vanishes as expected. If we integrate (6.22) in turn over all V_d, using spherical coördinates, we find that

$$\int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \, 1 \;=\; \frac{4}{3}\pi(2\varepsilon)^3 \int_0^{2\varepsilon} dr_d \, 4\pi r_d^2 \left\{1 - \frac{3}{2}\left(\frac{r_d}{2\varepsilon}\right) + \frac{1}{2}\left(\frac{r_d}{2\varepsilon}\right)^3\right\}$$

$$= \frac{1}{8}\left(\frac{4}{3}\pi(2\varepsilon)^3\right)^2, \tag{6.23}$$

which is the expected result—the factor of $1/8 = 1/2^3$ taking care of the "other half" of the factor of $2^6 = 64$ apparent increase in volume of the 6-dimensional integration space (the Jacobian providing the other factor of 1/8).

We defer a computation of the integral (6.21) to Section 6.4.5; we shall first need to derive integrals over V_d more complicated than (6.23).

6.4.4 Explicit outer integrals

While we have predicted that the $\boldsymbol{r}_s$-dependencies of the terms in our radiation reaction expressions will *only* be of one of the two forms (6.20) or (6.21), the $\boldsymbol{r}_d$-dependencies of the terms do not have such simplicity. In fact, they involve an almost arbitrary number of factors of $\boldsymbol{r}_d$, *as well as* factors r_d^n of the magnitude r_d of $\boldsymbol{r}_d$. (The latter can be understood in terms of the discussion of Section 6.4: they arise as the leading-order term of single powers of R, the magnitude of $\boldsymbol{R}$, which is also the negative of the value of the retarded time $t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}')$.)

Now, since the region of integration V_d *is* spherically symmetric, the angular and radial integrals *may* be decoupled by the use of spherical coördinates. We shall consider the angular integrations shortly. For the case where the integrand in question is *independent of* $\boldsymbol{r}_s$, we simply need to consider integrals of the form

$$\eta_m \equiv \frac{1}{4\pi}\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r' \, \frac{1}{r_d^m},$$

where we are here defining the symbols η_m, promised in Chapter 5. As with (6.23), these may be computed by elementary integration; by using the result (6.22), and simplifying somewhat, one finds that

$$\eta_m = \frac{72}{(2\varepsilon)^m(3-m)(4-m)(6-m)}\,\eta_0. \tag{6.24}$$

The result (6.24) is not intuitively obvious, and would have been perplexing if we had not explicitly outlined its method of derivation above. Of particular interest to our considerations, we have

$$\eta_1 = \frac{6}{5\varepsilon}\,\eta_0, \tag{6.25}$$

$$\eta_2 = \frac{9}{4\varepsilon^2}\,\eta_0, \tag{6.26}$$

$$\eta_3 = \infty. \tag{6.27}$$

The infiniteness of η_3 is simply understood: the Jacobian factor $4\pi r_d^2$ is, in this case, insufficient to overcome the factor of $1/r_d^3$ in the integrand. We shall examine the subtle consequences of this result in Section 6.7.

6.4.5 Quadratic inner integrals

We now consider the calculation of $\boldsymbol{r}_s$ integrals with two factors of $\boldsymbol{r}_s$ present, as promised in Section 6.4.3, *viz.*, those of the form (6.21).

We firstly note that, at each level of integration in (6.19), the domain of integration of x, y or z is from $-A$ to $+A$ (where A is some number), and any extra factors present are even functions of x, y or z. Thus, for integrands (6.21) where $i \neq j$, we find a vanishing integral on symmetry grounds.

For the case $i = j$, the circular symmetry of the expressions around the z-axis means that it suffices to compute the integral of z^2, as well as that of either x^2 *or* y^2; we choose to integrate y^2.

Firstly, for the integral of z^2, we use the results of Section 6.4.3 for the x and y integrals of unity, and find

$$\begin{aligned}\int_{V_s} d^3 r_s\, z^2 &= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \quad \pi z^2 \Big\{ 4\varepsilon^2 - (|z| + 2\varepsilon\alpha)^2 \Big\} \\ &= 2\pi \int_0^{2\varepsilon(1-\alpha)} dz \quad z^2 \Big\{ 4\varepsilon^2 - (|z| + 2\varepsilon\alpha)^2 \Big\}\end{aligned}$$

$$= \frac{4}{3}\pi(2\varepsilon)^5 \left\{ \frac{1}{5} - \frac{3}{4}\alpha + \alpha^2 - \frac{1}{2}\alpha^3 + \frac{1}{20}\alpha^5 \right\}. \qquad (6.28)$$

We note that, when $\alpha = 1$, (6.28) vanishes, as required. Turning now to the integral of y^2, we find

$$\begin{aligned}
\int_{V_s} d^3 r_s\, y^2 &= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \int_{-\rho(z,\alpha)}^{\rho(z,\alpha)} dy \quad 2y^2 \sqrt{\rho^2(z,\alpha) - y^2} \\
&= \int_{-2\varepsilon(1-\alpha)}^{2\varepsilon(1-\alpha)} dz \quad \frac{1}{4}\pi\rho^4(z,\alpha) \\
&= \frac{1}{2}\pi \int_0^{2\varepsilon(1-\alpha)} dz \ \left\{ 4\varepsilon^2 - (|z| + 2\varepsilon\alpha)^2 \right\}^2 \\
&= \frac{4}{3}\pi(2\varepsilon)^5 \left\{ \frac{1}{5} - \frac{3}{8}\alpha + \frac{1}{4}\alpha^3 - \frac{3}{40}\alpha^5 \right\}. \qquad (6.29)
\end{aligned}$$

Again, when $\alpha = 1$, (6.29) vanishes, as required. Also, for $\alpha = 0$, the results (6.28) and (6.29) are equal—as they should be, by symmetry, since for $\alpha = 0$ the region of integration is simply a sphere.

We now do away with our interim (x, y, z) set of axes, and rewrite the results (6.28) and (6.29) using only the components of the quantity $\boldsymbol{r}_d$. Clearly, we have

$$\begin{aligned}
\int_{V_s} d^3 r_s \ r_s^i r_s^j &= \frac{4}{3}\pi(2\varepsilon)^5 \left(\frac{r_d}{2\varepsilon}\right) \left\{ -\frac{3}{8} + \left(\frac{r_d}{2\varepsilon}\right) - \frac{3}{4}\left(\frac{r_d}{2\varepsilon}\right)^2 + \frac{1}{8}\left(\frac{r_d}{2\varepsilon}\right)^4 \right\} n_d^i n_d^j \\
&\quad + \frac{4}{3}\pi(2\varepsilon)^5 \left\{ \frac{1}{5} - \frac{3}{8}\left(\frac{r_d}{2\varepsilon}\right) + \frac{1}{4}\left(\frac{r_d}{2\varepsilon}\right)^3 - \frac{3}{40}\left(\frac{r_d}{2\varepsilon}\right)^5 \right\} \delta^{ij}. \quad (6.30)
\end{aligned}$$

To check that the calculation of the result (6.30) is in fact correct, we compute

$$\begin{aligned}
&\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r \le \varepsilon} d^3 r \int_{r' \le \varepsilon} d^3 r' \, r_s^i r_s^j \\
&= \frac{1}{8}\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \ r_s^i r_s^j \\
&= \eta_0 \, (2\varepsilon)^2 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-1} \int_{V_d} d^3 r_d \left(\frac{r_d}{2\varepsilon}\right) \left\{ -\frac{3}{8} + \left(\frac{r_d}{2\varepsilon}\right) - \frac{3}{4}\left(\frac{r_d}{2\varepsilon}\right)^2 + \frac{1}{8}\left(\frac{r_d}{2\varepsilon}\right)^4 \right\} n_d^i n_d^j
\end{aligned}$$

$$+\,\eta_0\,(2\varepsilon)^2\left(\frac{4}{3}\pi\varepsilon^3\right)^{-1}\!\int_{V_d}\! d^3r_d\left\{\frac{1}{5}-\frac{3}{8}\left(\frac{r_d}{2\varepsilon}\right)+\frac{1}{4}\left(\frac{r_d}{2\varepsilon}\right)^3-\frac{3}{40}\left(\frac{r_d}{2\varepsilon}\right)^5\right\}\delta^{ij}. \quad (6.31)$$

Now, as will be shown shortly, the angular integration of two factors of $\boldsymbol{n}_d$ yields

$$\int_{V_d} d^3r_d\; n_d^i n_d^j\, f(r_d) = \frac{1}{3}\delta^{ij}\!\int_{V_d} d^3r_d\, f(r_d);$$

thus, (6.31) yields

$$\begin{aligned}
&\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\!\int_{r\le\varepsilon} d^3r\int_{r'\le\varepsilon} d^3r'\; r_s^i r_s^j \\
&= \eta_0\,(2\varepsilon)^2\left(\frac{4}{3}\pi\varepsilon^3\right)^{-1}\!\int_0^{2\varepsilon} 4\pi r_d^2\,dr_d\left\{\frac{1}{5}-\frac{1}{2}\left(\frac{r_d}{2\varepsilon}\right)+\frac{1}{3}\left(\frac{r_d}{2\varepsilon}\right)^2-\frac{1}{30}\left(\frac{r_d}{2\varepsilon}\right)^5\right\} \\
&= \frac{2\varepsilon^2}{5}\eta_0 \\
&= \frac{1}{3}\eta_{-2}.
\end{aligned} \quad (6.32)$$

That this result, $\eta_{-2}/3$, is indeed correct follows from the fact that we could have alternatively computed it as the integral of $r_d^i r_d^j$; the normals $n_d^i n_d^j$ contribute the factor of $1/3$, and the magnitudes r_d^2 contribute the integral η_{-2}.

We now employ clairvoyance once again, and predict that the only integrals involving $r_s^i r_s^j$ that we shall have need to perform will be of the form

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\!\int_{r\le\varepsilon} d^3r\int_{r'\le\varepsilon} d^3r'\; r_d^{-m} r_s^2 n_s^i n_s^j\, f(\boldsymbol{n}_d), \quad (6.33)$$

where $m = 2$ or 3 *only*. While the integral over $\boldsymbol{n}_d$ cannot be performed without the explicit $f(\boldsymbol{n}_d)$ being supplied, we can nevertheless perform the complete $\boldsymbol{r}_s$ integral, as well as the integral over r_d, following the same method as above. When this is done for $m = 2$, one finds

$$\begin{aligned}
&\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\!\int_{r\le\varepsilon} d^3r\int_{r'\le\varepsilon} d^3r'\; r_d^{-2} r_s^2 n_s^i n_s^j\, f(\boldsymbol{n}_d) \\
&\qquad = \eta_0\int d^2n_d\left\{\frac{3}{2}\delta^{ij}-\frac{1}{2}n_d^i n_d^j\right\} f(\boldsymbol{n}_d)
\end{aligned} \quad (6.34)$$

where by the notation $\int d^2 n_d$ we mean the integral over the angular degrees of freedom of $\boldsymbol{r}_d$, but *divided by* 4π. (*E.g.*, $\int d^2 n_d\, 1 = 1$.) The effect of the result (6.34) may be equivalently incorporated into our calculations, in the following sections, by means of the prescription

$$r_d^{-2} r_s^2 n_s^i n_s^j \longrightarrow \frac{3}{2}\delta^{ij} - \frac{1}{2} n_d^i n_d^j, \tag{6.35}$$

followed by an $\boldsymbol{r}_d$ integration performed as if there had been *no* $\boldsymbol{r}_s$-dependence at all.

Because the result (6.35) has a direct and major influence on the final radiation equations of motion that will be obtained, it is important to cross-check it in some way. The most obvious method is to integrate it over all d^3r and d^3r':

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\le\varepsilon} d^3r \int_{r'\le\varepsilon} d^3r'\, r_d^{-2} r_s^2 n_s^i n_s^j = \left\{\frac{3}{2} - \frac{1}{2}\cdot\frac{1}{3}\right\}\eta_0\,\delta^{ij} = \frac{4}{3}\eta_0\,\delta^{ij}, \tag{6.36}$$

and then to note that, had we exchanged $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$ before performing the integral, the $n_d^i n_d^j$ factors would then trivially contribute a factor of $1/3$; this, together with (6.36), implies that

$$\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\le\varepsilon} d^3r \int_{r'\le\varepsilon} d^3r'\, r_d^{-2} r_s^2 = 4. \tag{6.37}$$

If we could prove the result (6.37) quickly and simply, from first principles, then we would be implicitly verifying the result (6.36), and hence (6.35). Unfortunately, the author has not been able to obtain any simple analytical computation of the integral (6.37), other than that used to derive (6.36). For this reason, a small and rudimentary computer program, CHECKRS, was written to perform the integral (6.37) numerically; the output is listed in Section G.8 of Appendix G. (The source code is only supplied with digital copies of this thesis; *see* Appendix G for details.) It will be noted that the numerically integrated result very quickly converges to the value 4; this verifies the integral (6.36), and strongly suggests that (6.35) is, indeed, correct.

The case of (6.33) with $m = 3$ is more subtle, again due to the fact that the radial r_d integral is, due to its inverse-cube nature, infinite. The term proportional to $n_d^i n_d^j$ in (6.33) avoids this fate (due to the fact that it contains an extra power of $r_d/2\varepsilon$ in all terms), but the term proportional to δ^{ij} does not. However, it will be found that, in fact, terms involving δ^{ij} will all *cancel* in our final expressions, *before* they need to be integrated; let us therefore simply write its integral, unperformed, as I_3^{ij}.

Performing, then, the integral of the $n_d^i n_d^j$ part explicitly, we find

$$\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-2} \int_{r \le \varepsilon} d^3 r \int_{r' \le \varepsilon} d^3 r' \, r_d^{-3} r_s^2 n_s^i n_s^j \, f(\boldsymbol{n}_d) \; = \; -\eta_1 \int d^2 n_d \, n_d^i n_d^j \, f(\boldsymbol{n}_d) + I_3^{ij}, \tag{6.38}$$

Again, the result (6.38) may be equivalently incorporated into our calculations by making the simple substitution

$$r_d^{-3} r_s^2 n_s^i n_s^j \longrightarrow I_3^{ij} - r_d^{-1} n_d^i n_d^j, \tag{6.39}$$

and then performing the $\boldsymbol{r}_d$ integration as if there had been no $\boldsymbol{r}_s$-dependence at all.

Unfortunately, it is not so simple to devise such a straightforward cross-check of (6.39) as was performed earlier for (6.35), due to the presence of the divergent integral I_3^{ij}. Instead, the author has thoroughly verified this result by directly comparing the term-by-term integrations of r_d^{-3} with those of r_d^0 (which latter have been verified above), which are of course in the ratio $(n+1)/8\varepsilon^3(n-2)$ if the overall integrand (including $4\pi r_d^2$ factor) has dependence r_d^n. (This can be seen quickly by integrating the polynomials from first principles, and noting that the extra factor of r_d^3 contributes $(2\varepsilon)^3$.)

6.4.6 Angular outer integrals

We now turn to the question of performing integrals over the angular coördinates $\boldsymbol{n}_d$ of $\boldsymbol{r}_d$, *viz.*, integrals of the form

$$\int d^2n_d \, f(\boldsymbol{n}_d). \tag{6.40}$$

Clearly, since the volume of integration of $\boldsymbol{r}_d$ is a simple sphere, we may separate the r_d and $\boldsymbol{n}_d$ integrals completely by employing spherical coördinates. (This was not possible for the $\boldsymbol{r}_s$ integrals above, since the volume V_s is not, in general, spherical.) Obviously, if the $f(\boldsymbol{n}_d)$ in (6.40) contains an odd number of factors of $\boldsymbol{n}_d$, the integral will vanish identically, by symmetry. We are thus left with integrals of the form

$$\begin{aligned} &\int d^2n_d \; n_d^i n_d^j, \\ &\int d^2n_d \; n_d^i n_d^j n_d^k n_d^l, \\ &\int d^2n_d \; n_d^i n_d^j n_d^k n_d^l n_d^m n_d^n, \end{aligned} \tag{6.41}$$

and so on. We shall at this point employ clairvoyance for the third and final time, and predict that the integrals (6.41) exhaust those that we shall need in the radiation reaction calculations. Now, consideration of three-space covariance alone requires the integrals (6.41) to be able to be written solely in terms of the three-covariant quantity δ^{ij}—there being no other covariant quantity available after the integration has been performed. This, together with due consideration of the symmetry of the expressions (6.41), already tells us that the answers must be of the form

$$\begin{aligned} \int d^2n_d \; n_d^i n_d^j &= \beta_2 \, \delta^{ij}, \\ \int d^2n_d \; n_d^i n_d^j n_d^k n_d^l &= \beta_4 \left\{ \delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} \right\}, \\ \int d^2n_d \; n_d^i n_d^j n_d^k n_d^l n_d^m n_d^n &= \beta_6 \left\{ \delta^{ij}\delta^{kl}\delta^{mn} + \delta^{ij}\delta^{km}\delta^{ln} + \delta^{ij}\delta^{kn}\delta^{lm} \right. \end{aligned}$$

$$+\delta^{ik}\delta^{jl}\delta^{mn}+\delta^{ik}\delta^{jm}\delta^{ln}+\delta^{ik}\delta^{jn}\delta^{lm}$$
$$+\delta^{il}\delta^{jk}\delta^{mn}+\delta^{il}\delta^{jm}\delta^{kn}+\delta^{il}\delta^{jn}\delta^{km}$$
$$+\delta^{im}\delta^{jk}\delta^{ln}+\delta^{im}\delta^{jl}\delta^{kn}+\delta^{im}\delta^{jn}\delta^{km}$$
$$+\delta^{in}\delta^{jk}\delta^{lm}+\delta^{in}\delta^{jl}\delta^{km}+\delta^{in}\delta^{jm}\delta^{kl}\Big\}, \quad (6.42)$$

where the constants β_{2m} must now be determined. To do so, it suffices to set all of the indices in the expressions (6.42) to the value z; noting that for $2m$ factors of $\boldsymbol{n}_d$ there are $(2m-1)!!$ terms in the symmetrised expansion (6.42), and using the angular part of spherical coördinates, we find

$$\int d^2n_d\ (n_d^z)^{2m} = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} d[\sin\theta]\sin^{2m}\theta = \frac{1}{2m+1};$$

hence,

$$\beta_{2m} = \frac{1}{(2m+1)!!}. \quad (6.43)$$

Explicitly,

$$\beta_0 = 1,$$
$$\beta_2 = \frac{1}{3},$$
$$\beta_4 = \frac{1}{15},$$
$$\beta_6 = \frac{1}{105}. \quad (6.44)$$

Thus, practically speaking, when one encounters an integrand with $2m$ factors of $\boldsymbol{n}_d$ present, one multiplies it by the corresponding β_{2m} from (6.44), and then replaces it with the $(2m-1)!!$ terms obtained by inserting Kronecker delta functions, in all possible ways, for the factors of $\boldsymbol{n}_d$. (In fact, the computer algebra library for the programs RADREACT and TEST3INT performs this computation *recursively*, and hence can handle an arbitrary number of normal factors, not just six.)

6.4.7 Is there an easier way?

It might be wondered whether the somewhat tortuous considerations of this section might be unnecessary if one were simply to *avoid* choosing the three-sphere as a region of integration for $\boldsymbol{r}$ altogether, and instead integrate over all space.

However, the problem would just be pushed somewhere else: we would require the use of some suitable non-trivial moment-density function $\rho_a(\boldsymbol{r})$ that ensures that the electromagnetic moments are confined to a volume of spatial extent of order ε. To maintain the simplicity of the centre-of-moment calculations, this moment density would be most simply chosen to be a function of radial distance only: $\rho = \rho(r)$. But then, in transforming to the new variables $\boldsymbol{r}_s$ and $\boldsymbol{r}_d$, the moment density expressions $\rho(r)$ and $\rho(r')$ would themselves be converted into hopelessly complicated functions of the vectors $\boldsymbol{r}_s$ and $\boldsymbol{r}_d$; the resulting integrals would be intractable. Furthermore, as already discussed in Section 6.3.2, we would then need to compute anew the static fields of a particle with such a modified moment density function, which may themselves be intractable.

Thus, the considerations of this sections *are* the simplest way that one can proceed, as far as the author can ascertain.

6.5 Retarded kinematical quantities

We shall now proceed to calculate the retarded kinematical quantities of the sending constituent $\boldsymbol{r}'$, that will be used to compute the retarded fields generated by the particle.

6.5.1 The retarded radius vector

The three-vector $\boldsymbol{R}$, of (6.5), is simply given by the three-displacement between the receiving constituent $\boldsymbol{r}$ at $t = 0$, and the generating constituent $\boldsymbol{r}'$

at $t = t_{\mathrm{ret}} \equiv -R$; thus,

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{z}_{r'}(t_{\mathrm{ret}}). \tag{6.45}$$

Using (3.23), we have

$$\begin{aligned}
\boldsymbol{z}_{r'}(t) \;=\; & \boldsymbol{r}' + \frac{1}{2}t^2\dot{\boldsymbol{v}} + \frac{1}{6}t^3\ddot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{24}t^4\dddot{\boldsymbol{v}} - \frac{1}{3}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& - \frac{1}{6}t^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}t^5\ddddot{\boldsymbol{v}} - \frac{1}{8}t^4(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} \\
& - \frac{1}{8}t^4(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{24}t^4(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} \\
& + \frac{1}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^6).
\end{aligned} \tag{6.46}$$

Thus, by (6.45),

$$\begin{aligned}
\boldsymbol{R} \;=\; & \boldsymbol{r} - \boldsymbol{r}' - \frac{1}{2}R^2\dot{\boldsymbol{v}} + \frac{1}{6}R^3\ddot{\boldsymbol{v}} + \frac{1}{2}R^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}R^4\dddot{\boldsymbol{v}} - \frac{1}{3}R^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& - \frac{1}{6}R^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}R^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}R^5\ddddot{\boldsymbol{v}} + \frac{1}{8}R^4(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} \\
& + \frac{1}{8}R^4(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{24}R^4(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}R^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} \\
& + \frac{1}{2}R^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}R^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^6).
\end{aligned} \tag{6.47}$$

6.5.2 Length of expressions

Because the expressions from this point become exceedingly long, we shall omit terms of lower orders in the explicit expressions listed here, where such an omission leads to a considerable shortening; the full expressions are given in Section G.6 of Appendix G.

However, the $\mathrm{O}(\varepsilon^n)$ notation included here still indicates the order retained in the *full expressions*, ignoring the fact that terms have been omitted for typographical sanity.

6.5.3 Final expression for the retarded radius vector

We now rewrite (6.47) in terms of the variables $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$ of Section 6.4:

$$\boldsymbol{R} = \boldsymbol{r}_d - \frac{1}{2}R^2\dot{\boldsymbol{v}} + \frac{1}{6}R^3\ddot{\boldsymbol{v}} - \frac{1}{4}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}R^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^6). \quad (6.48)$$

To eliminate R^2, R^3, R^4 and R^5, we firstly square equation (6.48) itself to get

$$\begin{aligned} R^2 &= r_d^2 - R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) + \frac{1}{4}R^4\dot{\boldsymbol{v}}^2 + \frac{1}{3}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) - \frac{1}{2}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 \\ &\quad + \frac{1}{2}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \cdots + \mathrm{O}(\varepsilon^7). \end{aligned}$$

Substituting this equation back into itself until R is eliminated from the right-hand side, we thus find that

$$\begin{aligned} R^2 &= r_d^2\Big\{1 - (\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 + \frac{1}{4}r_d^2\dot{\boldsymbol{v}}^2 + \frac{1}{3}r_d(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) + \frac{1}{2}(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\ &\quad + \cdots + \mathrm{O}(\varepsilon^5)\Big\}. \end{aligned} \quad (6.49)$$

Employing the unit vectors $\boldsymbol{n}_d$ and $\boldsymbol{n}_s$,

$$\begin{aligned} \boldsymbol{n}_d &\equiv \frac{\boldsymbol{r}_d}{r_d}, \\ \boldsymbol{n}_s &\equiv \frac{\boldsymbol{r}_s}{r_s}, \end{aligned} \quad (6.50)$$

to more explicitly exhibit the dimensionalities of the various terms in each expression, and using the binomial theorem on (6.49), we thus find

$$\boldsymbol{R} = r_d\Big\{\boldsymbol{n}_d - \frac{1}{2}r_d\dot{\boldsymbol{v}} + \frac{1}{6}r_d^2\ddot{\boldsymbol{v}} + \frac{1}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^5)\Big\}. \quad (6.51)$$

6.5.4 The retarded normal vector

The retarded field expressions of Chapter 5 do not explicitly use the three-vector $\boldsymbol{R}$; rather, they are somewhat simplified by using $\boldsymbol{n}$. We can compute

$\boldsymbol{n}$ via the inverse of (6.5), namely,

$$\boldsymbol{n} \equiv \frac{\boldsymbol{R}}{R}.$$

By using (6.51) and (6.49), one finds that

$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{n}_d - \frac{1}{2}r_d\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{6}r_d^2\ddot{\boldsymbol{v}} - \frac{1}{8}r_d^2\dot{\boldsymbol{v}}^2\boldsymbol{n}_d + \frac{1}{8}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d \\ &\quad - \frac{1}{6}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{4}r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^5).\end{aligned} \tag{6.52}$$

6.5.5 Retarded velocity and derivatives

Taking successive t-derivatives of (6.46), we have

$$\begin{aligned}\boldsymbol{v}_{r'}(t) &= t\dot{\boldsymbol{v}} + \frac{1}{2}t^2\ddot{\boldsymbol{v}} - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{6}t^3\dddot{\boldsymbol{v}} - t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad + t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{24}t^4\ddddot{\boldsymbol{v}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{6}t^3(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{1}{2}t^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} \\ &\quad + \mathrm{O}(\varepsilon^5),\end{aligned} \tag{6.53}$$

$$\begin{aligned}\dot{\boldsymbol{v}}_{r'}(t) &= \dot{\boldsymbol{v}} + t\ddot{\boldsymbol{v}} - (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2\dddot{\boldsymbol{v}} - 2t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - t(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\ &\quad + \frac{1}{6}t^3\ddddot{\boldsymbol{v}} - \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - \frac{3}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{3}{2}t^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad + 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^4),\end{aligned} \tag{6.54}$$

$$\begin{aligned}\ddot{\boldsymbol{v}}_{r'}(t) &= \ddot{\boldsymbol{v}} + t\dddot{\boldsymbol{v}} - 2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - (\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2\ddddot{\boldsymbol{v}} - 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - 3t(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\ &\quad - t(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 3t\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad + \mathrm{O}(\varepsilon^3).\end{aligned} \tag{6.55}$$

Using the fact that $t_{\text{ret}} \equiv -R$, and employing the expression (6.49), we thus have

$$\begin{aligned}\boldsymbol{v}_{\text{ret}} &= -r_d\dot{\boldsymbol{v}} + \frac{1}{2}r_d^2\ddot{\boldsymbol{v}} + \frac{1}{2}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{6}r_d^3\dddot{\boldsymbol{v}} - \frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{1}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\ &\quad + \frac{1}{12}r_d^3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^2 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{4}r_d^2 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{4}r_d r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\ &\quad + \cdots + \mathrm{O}(\varepsilon^5), \end{aligned} \tag{6.56}$$

$$\begin{aligned}\dot{\boldsymbol{v}}_{\text{ret}} &= \dot{\boldsymbol{v}} - r_d\ddot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_d^2\dddot{\boldsymbol{v}} - \frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\ &\quad + \frac{1}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{2}r_d r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{1}{2}r_d r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^4), \end{aligned} \tag{6.57}$$

$$\begin{aligned}\ddot{\boldsymbol{v}}_{\text{ret}} &= \ddot{\boldsymbol{v}} - r_d\dddot{\boldsymbol{v}} + r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad + \cdots + \mathrm{O}(\varepsilon^3). \end{aligned} \tag{6.58}$$

6.5.6 Constituent retarded FitzGerald spins

For purposes of calculating the retarded fields, we need to use the FitzGerald spin vector, $\boldsymbol{\sigma}'_{r'}(t)$, and its derivatives. Using (6.56) and (6.49) in (3.24) and its derivatives, we find

$$\begin{aligned}\boldsymbol{\sigma}'_{\text{ret}} &= \boldsymbol{\sigma} - r_d\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^2\ddot{\boldsymbol{\sigma}} - \frac{1}{2}r_d^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\ &\quad + \cdots + \mathrm{O}(\varepsilon^5) \end{aligned} \tag{6.59}$$

$$\begin{aligned}\dot{\boldsymbol{\sigma}}'_{\text{ret}} &= \dot{\boldsymbol{\sigma}} - r_d\ddot{\boldsymbol{\sigma}} + r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\ &\quad + \cdots + \mathrm{O}(\varepsilon^4), \end{aligned} \tag{6.60}$$

$$\ddot{\boldsymbol{\sigma}}'_{\text{ret}} = \ddot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^3). \tag{6.61}$$

6.5.7 Computer algebra

All of the quantities listed to this point were originally calculated by hand, by the author, to the full order of ε indicated. It should be apparent, upon reflection of the lengthy nature of the explicit expressions listed in Appendix G, why it was not feasible to continue to compute such expressions by hand.

Thus, the author decided to investigate methods of calculating and verifying the algebra by computational means. A description of this investigation is given in Section G.2 of Appendix G.

From this point, the expressions listed in Section G.6 were calculated to *one lower* order by hand; the computer algebra programs verified these lower-order results, as well as providing the next order of terms. Only one error was found in the manually-computed expressions, that had not yet been used for any consequential computations. Once the error in the manual expression had been located and corrected, the computer algebra program was used to verify the electric charge field results through to completion (which had also been completed, manually), and to compute, for the first time, the dipole field and radiation reaction results.

6.5.8 Gamma factor

Returning, now, to the algebraic derivation, we note that two terms in the retarded field expressions of Chapter 5 involve the Lorentz gamma factor γ_{ret}; however, it will be noted that, in both cases, it appears in the form

$$\gamma_{\text{ret}}^{-2} \equiv 1 - \boldsymbol{v}_{\text{ret}}^2.$$

To compute γ_{ret}^{-2}, we use (6.56); the result (to the lower order of ε actually required for this particular quantity) is

$$\gamma_{\text{ret}}^{-2} = 1 - r_d^2 \dot{\boldsymbol{v}}^2 + r_d^3 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) + r_d^2 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) + \cdots + \mathrm{O}(\varepsilon^6). \qquad (6.62)$$

6.5.9 Modified retarded normal vectors

We now compute the quantities $\boldsymbol{n}'$ and $\boldsymbol{n}''$, introduced in Chapter 5 to simplify the retarded field expressions:

$$\begin{aligned}\boldsymbol{n}' &\equiv \boldsymbol{n} - \boldsymbol{v}_{\text{ret}},\\ \boldsymbol{n}'' &\equiv \boldsymbol{n}' - \boldsymbol{v}_{\text{ret}}\times(\boldsymbol{n}\times\boldsymbol{v}_{\text{ret}}).\end{aligned}$$

Using (6.52) and (6.56), we have

$$\begin{aligned}\boldsymbol{n}' &= \boldsymbol{n}_d + \frac{1}{2}r_d\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{3}r_d^2\ddot{\boldsymbol{v}} - \frac{1}{8}r_d^2\dot{\boldsymbol{v}}^2\boldsymbol{n}_d + \frac{1}{8}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\ &\quad - \frac{1}{6}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_d r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\ &\quad + \cdots + \mathrm{O}(\varepsilon^5), \end{aligned} \tag{6.63}$$

and

$$\begin{aligned}\boldsymbol{n}'' &= \boldsymbol{n}_d + \frac{1}{2}r_d\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{3}r_d^2\ddot{\boldsymbol{v}} - \frac{9}{8}r_d^2\dot{\boldsymbol{v}}^2\boldsymbol{n}_d + r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\ &\quad + \frac{1}{8}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d - \frac{1}{6}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_d r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\ &\quad - \frac{1}{4}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \cdots + \mathrm{O}(\varepsilon^5). \end{aligned} \tag{6.64}$$

6.5.10 Retarded Doppler factor

We now compute the expression for the retarded Doppler factor, κ, introduced in Chapter 5:

$$\kappa \equiv \frac{1}{1-(\boldsymbol{v}\cdot\boldsymbol{n})}.$$

One finds

$$\begin{aligned}\kappa &= 1 - r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}r_d^2\dot{\boldsymbol{v}}^2 + \frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}}) + \frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\\ &\quad + \frac{1}{2}r_d r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) + \cdots + \mathrm{O}(\varepsilon^5). \end{aligned} \tag{6.65}$$

6.5.11 Final redshift expression

Finally, from the definition (3.21), we have

$$\lambda = 1 + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) + \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}). \tag{6.66}$$

6.5.12 Retarded field expressions

We can now use the preceding expressions to compute the various retarded fields of the particle, generated by the sending constituent $\boldsymbol{r}'$ and received by the receiving constituent $\boldsymbol{r}$.

It will be noted that, in all cases, the "accelerative redshift" factor $\lambda(\boldsymbol{r})$—that relates the constituent proper-time derivatives to body proper-time derivatives—will be multiplied in as the final step of the self-interaction calculations. Most of these calculations will involve the self-generated electric and magnetic fields as a simple factor; for these calculations, it is convenient to compute λ times the electric or magnetic field from the outset.

On the other hand, the self-interaction terms involving spatial *gradients* of the electromagnetic fields must be treated more carefully. Here, we must note that the operations of multiplying by $\lambda(\boldsymbol{r})$, and that of taking the spatial derivative ∇_r in $\boldsymbol{r}$-space, do *not* commute, because $\lambda(\boldsymbol{r})$ of course has an explicit dependence on $\boldsymbol{r}$. Thus, we may *not* compute (for example)

$$(\boldsymbol{\sigma} \cdot \nabla) \lambda(\boldsymbol{r}) \boldsymbol{E}(\boldsymbol{r});$$

we *must*, rather, compute

$$\lambda(\boldsymbol{r}) (\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{E}(\boldsymbol{r}).$$

The explicit expressions for the retarded self-fields of the particle are lengthy; we list them explicitly in Section G.6.19 of Appendix G.

6.6 Divergence of the point particle fields

In this section, we compute the spatial divergence of the fields generated by a pointlike particle, in the vicinity of the particle's worldline, by analysing the explicit expressions listed in Section G.6.19.

Our reasons for carrying out such a procedure are threefold:

Firstly, it will provide an explicit check of the veracity of the self-field expressions computed in the previous section, since, from Maxwell's equations, we know in advance what the divergence of the fields should be: the source expressions are simple at the instant that the particle is at rest.

Secondly, such a computation provides the concrete basis on which to add the extra contact field required for a magnetic dipole, over and above that of the dual of the electric dipole field, in the case of arbitrary motion of the particle. (This task was performed, on the basis of the results in this section, in Section 5.5.15.)

Thirdly, the method used to regularise the point particle expressions provides us with a suitable framework for carrying out, in the following sections, subtle but important integrations of divergent expressions in the radiation reaction calculations.

We begin, in Section 6.6.1, by explaining the particular regularisation procedure that we will use for the divergent point particle fields expressions; the explicit substitutions required are detailed in Sections 6.6.2 and 6.6.3. Various spatial gradients of these basic quantities are computed in Sections 6.6.4 and 6.6.5. These results are then used in Sections 6.6.6 and 6.6.7 to carefully examine the spatial gradients of the monopolar inverse-square and dipolar inverse-cube fields, and again in Section 6.6.8 for various inverse-*square* terms in the retarded dipole fields of Section G.6.19.

6.6.1 Regularisation procedure

Because the generated electromagnetic fields diverge at the worldline of a pointlike particle, we clearly need to "regularise" the field expressions, *i.e.*, modify the problem slightly so that all expressions are finite, and then take the point limit *after* we have computed gradients.

One possible approach would be to analyse the fields of the extended rigid body—as an expansion in ε, at any rate—by integrating the fields $\boldsymbol{E}^a_n(\boldsymbol{r}_d, \boldsymbol{r}_s, \varepsilon)$ and $\boldsymbol{B}^a_n(\boldsymbol{r}_d, \boldsymbol{r}_s, \varepsilon)$ over the three-sphere of integration of $\boldsymbol{r}'$, *i.e.*, by performing an integration over the *generating* volume only. But then we would be performing *no* integral over $\boldsymbol{r}$ itself; as a result, a simple integration in $\boldsymbol{r}_d$–$\boldsymbol{r}_s$ space would no longer be possible. Indeed, the author has not been able to find any tractable way of computing the integrals in this way.

Instead of this, we will instead consider the expressions for a *point* particle from the outset, and employ a subtle limiting procedure to obtain our results. Clearly, we may effectively shrink our body to a point by setting $\boldsymbol{r}' = \boldsymbol{0}$ in all expressions, and then examining how the fields behave around the position $\boldsymbol{r} = \boldsymbol{0}$. The quantity ε is now first order in $\boldsymbol{r}$ alone; we shall find that the resulting expressions do, in fact, possess sufficient orders of ε for us to take divergences, *etc.*, of the self-fields around the origin of $\boldsymbol{r}$—the position of the point particle at $t = 0$.

There are a number of methods that one can employ to regularise mathematical expressions around $\boldsymbol{r} = \boldsymbol{0}$, but—to the author, at least—the most conceptually straightforward is to *append an extra (Euclidean) dimension to $\boldsymbol{r}$-space*. In other words, we consider a new, Euclidean *four*-dimensional space, referred to as *appended space* or *$\tilde{\boldsymbol{r}}$-space*, possessing explicit Cartesian position coördinates

$$\tilde{\boldsymbol{r}} \equiv (w, x, y, z). \tag{6.67}$$

The presence of the extra dimension w generates a continuous range of three-dimensional subspaces $\boldsymbol{r}_w$ in $\tilde{\boldsymbol{r}}$-space; we consider the value of w to be a "godlike" parameter, *i.e.*, one whose value is set by the author, but, once set, cannot be changed by any of the operators in the equations being considered. Thus, we immediately rule out the operator ∂_w, the generator of w-translations, from being able to take any part in our considerations. The gradient operator is therefore still only a three-vector, namely, ∇; there is *no* operator $\tilde{\nabla}$ in $\tilde{\boldsymbol{r}}$-space.

Quantities are computed in $\tilde{\boldsymbol{r}}$-space as they are in $\boldsymbol{r}$-space, except that, where necessary, the appended dimension w is discarded, when a purely three-vector quantity is required; this is deemed to occur implicitly if an operation being performed is an explicit three-vector operation, *e.g.*, dot-products and cross-products. We then consider the resulting expressions in the limit $w \to 0$, *after* all necessary mathematical manipulations have been performed.

It should be noted that the process employed here is essentially a rigorous generalisation of the "a-potential" trick used by Jackson [113, Sec. 1.7].

6.6.2 Radial magnitude

The radial magnitude $\tilde{r}$ of $\tilde{\boldsymbol{r}}$ is given by the standard Euclidean result:

$$\tilde{r} \equiv (\tilde{\boldsymbol{r}}^2)^{1/2} \equiv (w^2 + x^2 + y^2 + z^2)^{1/2}. \tag{6.68}$$

We deem that, in extending divergent three-space expressions into appended space, the radial magnitude r is to be replaced by $\tilde{r}$ wherever it appears.

6.6.3 Directional normals

From the definitions (6.67) and (6.68) for the appended position vector and radial magnitude in $\tilde{\boldsymbol{r}}$-space, we can construct a unit vector $\tilde{\boldsymbol{n}}$:

$$\tilde{\boldsymbol{n}} \equiv \frac{\tilde{\boldsymbol{r}}}{\tilde{r}}. \tag{6.69}$$

Clearly, $\tilde{\boldsymbol{n}}^2 = 1$.

We must be careful in our notation, however, when discarding the w-component of $\tilde{\boldsymbol{n}}$: we write

$$\boldsymbol{n}_{\tilde{r}} \equiv \frac{\boldsymbol{r}}{\tilde{r}}. \tag{6.70}$$

The reason that we need to be careful is that this new three-vector, $\boldsymbol{n}_{\tilde{r}}$, is *not* equal to the simple three-vector $\boldsymbol{n}$:

$$\boldsymbol{n} \equiv \frac{\boldsymbol{r}}{r}; \tag{6.71}$$

the subscript $_{\tilde{r}}$ in (6.70) denotes this distinction; on the other hand, the lack of a tilde on $\boldsymbol{n}_{\tilde{r}}$ denotes that it *is* in fact a three-vector. Note that $\boldsymbol{n}_{\tilde{r}}$ does *not*, however, have unit magnitude:

$$\boldsymbol{n}_{\tilde{r}}^2 = \frac{r^2}{\tilde{r}^2} \neq 1. \tag{6.72}$$

We deem that, in extending divergent three-space expressions into appended space, the directional normal $\boldsymbol{n}$ is to be replaced by $\tilde{\boldsymbol{n}}$ wherever it appears. However, since the resulting $\tilde{\boldsymbol{n}}$ will always be either dot- or cross-producted into another three-vector, or have its appended w-component discarded for a three-vector result, it is simpler to consider the replacement as being $\boldsymbol{n} \rightarrow \boldsymbol{n}_{\tilde{r}}$ from the outset.

6.6.4 Radial magnitude gradients

We now compute the three-gradient of the radial magnitude $\tilde{r}$ in $\tilde{\boldsymbol{r}}$-space:

$$\begin{aligned} \nabla \tilde{r} &\equiv \nabla (w^2 + x^2 + y^2 + z^2)^{1/2} \\ &= \frac{(x, y, z)}{(w^2 + x^2 + y^2 + z^2)^{1/2}} \\ &= \frac{\boldsymbol{r}}{\tilde{r}}; \end{aligned} \tag{6.73}$$

hence,

$$\nabla \tilde{r} = \boldsymbol{n}_{\tilde{r}}. \tag{6.74}$$

Using the chain rule on (6.74), we find

$$\nabla \frac{1}{\tilde{r}^m} = -\frac{m\boldsymbol{n}_{\tilde{r}}}{\tilde{r}^{m+1}}. \tag{6.75}$$

6.6.5 Directional normal gradients

We now compute the gradient $\partial_i n_{\tilde{r}j}$:

$$\begin{aligned} \partial_i n_{\tilde{r}j} &\equiv \partial_i \frac{r_j}{\tilde{r}} \\ &= \frac{1}{\tilde{r}}\partial_i r_j + r_j \partial_i \frac{1}{\tilde{r}} \\ &= \frac{\delta_{ij}}{\tilde{r}} - \frac{n_{\tilde{r}i} r_j}{\tilde{r}^2}; \end{aligned}$$

hence,

$$\partial_i n_{\tilde{r}j} = \frac{\delta_{ij} - n_{\tilde{r}i} n_{\tilde{r}j}}{\tilde{r}}. \tag{6.76}$$

By contracting (6.76) with δ_{ij}, we immediately find

$$(\nabla \cdot \boldsymbol{n}_{\tilde{r}}) = \frac{1}{\tilde{r}} \left\{ 3 - \frac{r^2}{\tilde{r}^2} \right\}; \tag{6.77}$$

by contracting it with ε_{ijk}, we find

$$\nabla \times \boldsymbol{n}_{\tilde{r}} = \boldsymbol{0}; \tag{6.78}$$

and by contracting it with an arbitrary three-vector a_i we find

$$(\boldsymbol{a} \cdot \nabla)\boldsymbol{n}_{\tilde{r}} = \frac{\boldsymbol{a} - (\boldsymbol{n}_{\tilde{r}} \cdot \boldsymbol{a})\boldsymbol{n}_{\tilde{r}}}{\tilde{r}}. \tag{6.79}$$

We further note that

$$\begin{aligned} \nabla(\boldsymbol{n}_{\tilde{r}} \cdot \boldsymbol{a}) &\equiv (\boldsymbol{a} \cdot \nabla)\boldsymbol{n}_{\tilde{r}} + \boldsymbol{a} \times (\nabla \times \boldsymbol{n}_{\tilde{r}}) \\ &= \frac{\boldsymbol{a} - (\boldsymbol{n}_{\tilde{r}} \cdot \boldsymbol{a})\boldsymbol{n}_{\tilde{r}}}{\tilde{r}} + \boldsymbol{a} \times \boldsymbol{0}; \end{aligned}$$

hence,

$$\nabla(\boldsymbol{n}_{\tilde{r}} \cdot \boldsymbol{a}) = \frac{\boldsymbol{a} - (\boldsymbol{n}_{\tilde{r}} \cdot \boldsymbol{a})\boldsymbol{n}_{\tilde{r}}}{\tilde{r}}. \tag{6.80}$$

Taking the dot-product of (6.80) with another arbitary vector $\boldsymbol{b}$, we find

$$(\boldsymbol{b}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a}) = \frac{(\boldsymbol{a}\cdot\boldsymbol{b}) - (\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{\tilde{r}}. \tag{6.81}$$

Finally, in the case when $\boldsymbol{b} = \boldsymbol{n}_{\tilde{r}}$, we find

$$(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a}) = \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a}) - \boldsymbol{n}_{\tilde{r}}^2(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{\tilde{r}};$$

employing (6.72), we thus find that

$$(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a}) = \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{\tilde{r}}\left\{1 - \frac{r^2}{\tilde{r}^2}\right\}. \tag{6.82}$$

6.6.6 The monopolar inverse-square field

We now consider carefully the monopolar inverse-square field, employing the $\tilde{\boldsymbol{r}}$-space:

$$\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2}. \tag{6.83}$$

Our purposes in doing so are twofold. Firstly, we shall find that the $\tilde{\boldsymbol{r}}$-space provides a quick yet powerful way of divining the delta-function divergence of (6.83), *without* having to employ any integral theorems, or resorting to Poisson's equation for the scalar potential; the results found will also be of use in the following sections. Secondly, we shall show that the process of employing the $\tilde{\boldsymbol{r}}$-space for the monopolar field, and then computing the gradient to obtain the dipolar field, is completely equivalent to performing these two processes in the reverse order.

We first consider the spatial gradient of (6.83) in an arbitrary direction:

$$\begin{aligned}\partial_i \frac{n_{\tilde{r}j}}{4\pi\tilde{r}^2} &= \frac{1}{4\pi\tilde{r}^2}\partial_i n_{\tilde{r}j} + \frac{n_{\tilde{r}j}}{4\pi}\partial_i\frac{1}{\tilde{r}^2} \\ &= \frac{\delta_{ij} - n_{\tilde{r}i}n_{\tilde{r}j}}{4\pi\tilde{r}^3} - \frac{2n_{\tilde{r}i}n_{\tilde{r}j}}{4\pi\tilde{r}^3};\end{aligned}$$

thus,

$$\partial_i \frac{n_{\tilde{r}j}}{4\pi\tilde{r}^2} = \frac{\delta_{ij} - 3n_{\tilde{r}i}n_{\tilde{r}j}}{4\pi\tilde{r}^3}. \tag{6.84}$$

Our first use of (6.84) is to compute the divergence of the monopolar inverse-square field; we already know, in advance, that this should be a delta-function at the origin (*i.e.*, a point monopole). Contracting (6.84) with δ_{ij}, we find

$$\begin{aligned}\nabla\cdot\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} &\equiv \delta_{ij}\partial_i\frac{n_{\tilde{r}j}}{4\pi\tilde{r}^2}\\ &= \delta_{ij}\frac{\delta_{ij}-3n_{\tilde{r}i}n_{\tilde{r}j}}{4\pi\tilde{r}^3}\\ &= \frac{3-3\boldsymbol{n}_{\tilde{r}}^2}{4\pi\boldsymbol{n}_{\tilde{r}}^3};\end{aligned}$$

using (6.72), we thus find

$$\nabla\cdot\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} = \frac{3}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}. \qquad (6.85)$$

To make sense of this result, we rewrite $\tilde{r}^2$ in terms of r and w:

$$\tilde{r}^2 \equiv r^2 + w^2;$$

we then find

$$\frac{3}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\} = \frac{3w^2}{4\pi(r^2+w^2)^{5/2}}, \qquad (6.86)$$

Now, it will be noted that, for $r > 0$, the expression (6.86) vanishes in the limit $w \to 0$:

$$\left.\frac{3w^2}{4\pi(r^2+w^2)^{5/2}}\right|_{r>0,\,w\to 0} \longrightarrow \frac{3\cdot 0^2}{4\pi(r^2+0^2)^{5/2}} = 0.$$

On the other hand, for $r = 0$ but w finite, we have

$$\left.\frac{3w^2}{4\pi(r^2+w^2)^{5/2}}\right|_{r=0,\,w>0} = \frac{3w^2}{4\pi(0^2+w^2)^{5/2}} = \frac{3}{4\pi w^3};$$

thus, the value of the expression (6.86) *diverges cubically* at $r = 0$, in the limit $w \to 0$. These two properties of (6.86) lead us to conclude that it is, in fact, proportional to a three-dimensional Dirac delta-function in $\boldsymbol{r}$-space. To determine the precise coëfficient of the delta-function, we integrate the expression (6.86), for arbitrary w, over all three-space:

$$\int d^3r \, \frac{3w^2}{4\pi(r^2+w^2)^{5/2}} = \int_0^\infty 4\pi r^2 \, dr \, \frac{3w^2}{4\pi(r^2+w^2)^{5/2}};$$

using the change of variable

$$u \equiv \frac{r}{\tilde{r}} \equiv \frac{r}{(r^2+w^2)^{1/2}}, \tag{6.87}$$

whence

$$d_r u = \frac{w^2}{(r^2+w^2)^{3/2}},$$

we find that

$$\int d^3r \, \frac{3w^2}{4\pi(r^2+w^2)^{5/2}} = 3\int_0^1 du \, u^2 = 1;$$

hence,

$$\lim_{w\to 0} \frac{3}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\} \equiv \delta(\boldsymbol{r}). \tag{6.88}$$

Returning to (6.85), the identification (6.88) means that

$$\nabla\cdot \frac{\boldsymbol{n}}{4\pi r^2} = \delta(\boldsymbol{r}), \tag{6.89}$$

the elementary result (where we have discarded the tilde notation in the final expression).

We now return to the general expression (6.84), and consider contracting it with an arbitrary three-vector $\boldsymbol{a}$. We find

$$\begin{aligned}(\boldsymbol{a}\cdot\nabla)\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} &\equiv a_i \frac{\delta_{ij}-3n_{\tilde{r}i}n_{\tilde{r}j}}{4\pi\tilde{r}^3} \\ &= \frac{a_j - 3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})n_{\tilde{r}j}}{4\pi\tilde{r}^3};\end{aligned}$$

hence,

$$-(\boldsymbol{a}\cdot\nabla)\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} = \frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})\boldsymbol{n}_{\tilde{r}} - \boldsymbol{a}}{4\pi\tilde{r}^3}. \tag{6.90}$$

The significance of this result is that it is precisely what we would obtain by replacing $\boldsymbol{n}$ and r in the static field result of Chapter 5 by the appended quantities $\boldsymbol{n}_{\tilde{r}}$ and $\tilde{r}$; the result is thus the same no matter which way we choose to proceed, as we should demand on the basis of consistency.

6.6.7 The dipolar inverse-cube field

We now provide another brief illustration of utility and simplicity of $\tilde{\boldsymbol{r}}$-space by employing it in the computation of the divergence of the inverse-cube static dipole field, namely,

$$\nabla\cdot\,\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})\boldsymbol{n}_{\tilde{r}}-\boldsymbol{a}}{4\pi\tilde{r}^3}.$$

We find

$$\begin{aligned}
\nabla\cdot\,\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})\boldsymbol{n}_{\tilde{r}}-\boldsymbol{a}}{4\pi\tilde{r}^3} &= \frac{3}{4\pi\tilde{r}^3}(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})+\frac{3}{4\pi}(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)\frac{1}{\tilde{r}^3}\\
&\quad+\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^3}(\nabla\cdot\boldsymbol{n}_{\tilde{r}})-\frac{1}{4\pi}(\boldsymbol{a}\cdot\nabla)\frac{1}{\tilde{r}^3}\\
&= \frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}-\frac{9(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\frac{r^2}{\tilde{r}^2}\\
&\quad+\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{3-\frac{r^2}{\tilde{r}^2}\right\}+\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4};
\end{aligned}$$

hence,

$$\nabla\cdot\,\frac{3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})\boldsymbol{n}_{\tilde{r}}-\boldsymbol{a}}{4\pi\tilde{r}^3}=\frac{15(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}. \qquad (6.91)$$

To make sense of this result, we take the gradient in the direction of $\boldsymbol{a}$ of the regularised representation (6.88) of the delta-function $\delta(\boldsymbol{r})$:

$$\begin{aligned}
(\boldsymbol{a}\cdot\nabla)\delta(\boldsymbol{r}) &\equiv (\boldsymbol{a}\cdot\nabla)\frac{3}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}\\
&= \frac{3}{4\pi}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}(\boldsymbol{a}\cdot\nabla)\frac{1}{\tilde{r}^3}-\frac{3}{4\pi\tilde{r}^3}(\boldsymbol{a}\cdot\nabla)\frac{r^2}{\tilde{r}^2}
\end{aligned}$$

$$= -\frac{9(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{1-\frac{r^2}{\tilde{r}^2}\right\} - \frac{6(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}$$
$$= -\frac{15(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^4}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}.$$

We thus conclude that

$$\nabla\cdot\frac{3(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{n}-\boldsymbol{a}}{4\pi r^3} = -(\boldsymbol{a}\cdot\nabla)\delta(\boldsymbol{r}), \tag{6.92}$$

as we would expect for the dipolar static field from first principles. (The minus sign here can be understood by recalling that the derivative of the one-dimensional delta-function $\delta(t)$ is *positive* for $t = 0^-$ and *negative* for $t = 0^+$; an infinitesimal dipole has its charges reversed in sign from this.)

6.6.8 The dipolar inverse-square terms

In Section 6.6.6 above, we showed, with the help of $\tilde{\boldsymbol{r}}$-space, how the divergence of the monopolar inverse-square field yields the desired delta-function source expression; in Section 6.6.7, the same considerations showed how the dipolar inverse-cube field yields a source expression which is the gradient of a delta-function in the direction of the dipole moment. We know that, at the moment that the particle is at rest, these contributions exhaust the source terms that should be present in Maxwell's equations: the divergences of all of the remaining terms in the point-particle field expressions should vanish.

Clearly, simple dimensional considerations reveal that, for field expressions of order r^{-1} or higher, there will be an insufficient number of inverse powers of w to yield any delta-function contributions to their divergence. However, regular functions of r may, of course, appear. The computer algebra program RADREACT, of Appendix G, computes these regular divergences explicitly, and finds that, when added together, they vanish, as expected.

On the other hand, there are also four terms in $4\pi\boldsymbol{E}^d_{\text{point}}$ and one term in $4\pi\boldsymbol{B}^d_{\text{point}}$ that are inverse-square; we need to verify that these terms do *not* lead to delta-function contributions to the divergence. (The regular functions that are part of their divergence are also checked explicitly with the computer algebra program in Appendix G.)

We can treat the lead term of $4\pi\boldsymbol{B}^d_{\rm point}$ most quickly:

$$\begin{aligned}\nabla\cdot\frac{\boldsymbol{n}_{\tilde r}\times\boldsymbol{a}}{4\pi\tilde r^2} &= \frac{1}{4\pi}\boldsymbol{n}_{\tilde r}\times\boldsymbol{a}\cdot\nabla\frac{1}{\tilde r^2}+\frac{1}{4\pi\tilde r^2}\boldsymbol{a}\cdot\nabla\times\boldsymbol{n}_{\tilde r}\\ &= -\frac{2}{4\pi}\boldsymbol{n}_{\tilde r}\times\boldsymbol{a}\cdot\frac{\boldsymbol{n}_{\tilde r}}{\tilde r^3}+\frac{1}{4\pi\tilde r^2}\boldsymbol{a}\cdot\boldsymbol{0}\\ &= 0.\end{aligned}$$

We now turn to the four inverse-square terms in $4\pi\boldsymbol{E}^d_{\rm point}$. The divergence of the first two are of the form

$$\nabla\cdot\frac{(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})\boldsymbol{b}-(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{b})\boldsymbol{a}}{4\pi\tilde r^2}. \tag{6.93}$$

Computing the first term, we have

$$\begin{aligned}\nabla\cdot\frac{(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})\boldsymbol{b}}{4\pi\tilde r^2} &= \frac{1}{4\pi\tilde r^2}(\boldsymbol{b}\cdot\nabla)(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})+\frac{1}{4\pi}(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})(\boldsymbol{b}\cdot\nabla)\frac{1}{\tilde r^2}\\ &= \frac{(\boldsymbol{a}\cdot\boldsymbol{b})-(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{b})}{4\pi\tilde r^3}-\frac{2(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{b})}{4\pi\tilde r^3};\end{aligned}$$

hence,

$$\nabla\cdot\frac{(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})\boldsymbol{b}}{4\pi\tilde r^2}=\frac{(\boldsymbol{a}\cdot\boldsymbol{b})-3(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{b})}{4\pi\tilde r^3}. \tag{6.94}$$

Since this is symmetrical in $\boldsymbol{a}$ and $\boldsymbol{b}$, we thus conclude that the divergence (6.93) vanishes:

$$\nabla\cdot\frac{(\boldsymbol{n}\cdot\boldsymbol{a})\boldsymbol{b}-(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{a}}{4\pi r^2}=0, \tag{6.95}$$

where we drop the tildes in this final expression.

The divergence of the second pair of terms is of the form

$$\nabla\cdot\frac{(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde r}-3(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde r}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde r}}{4\pi\tilde r^2}. \tag{6.96}$$

The first term gives

$$\nabla\cdot\,\frac{(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} \equiv (\boldsymbol{a}\cdot\boldsymbol{b})\,\nabla\cdot\,\frac{\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2};$$

hence,

$$\nabla\cdot\,\frac{(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} = \frac{3(\boldsymbol{a}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}. \tag{6.97}$$

The second term of (6.96) gives

$$\begin{aligned}
\nabla\cdot\,\frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} &= \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^2}(\nabla\cdot\boldsymbol{n}_{\tilde{r}}) + \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi}(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)\frac{1}{\tilde{r}^2} \\
&\quad + \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})}{4\pi\tilde{r}^2}(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b}) + \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^2}(\boldsymbol{n}_{\tilde{r}}\cdot\nabla)(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a}) \\
&= \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{3-\frac{r^2}{\tilde{r}^2}\right\} - \frac{2(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\frac{r^2}{\tilde{r}^2} \\
&\quad + \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\} + \frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\};
\end{aligned}$$

hence,

$$\nabla\cdot\,\frac{(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} = \frac{5(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}. \tag{6.98}$$

Using (6.97) and (6.98) in (6.96), we thus find that

$$\nabla\cdot\,\frac{(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}} - 3(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})\boldsymbol{n}_{\tilde{r}}}{4\pi\tilde{r}^2} = 3\,\frac{(\boldsymbol{a}\cdot\boldsymbol{b}) - 5(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})}{4\pi\tilde{r}^3}\left\{1-\frac{r^2}{\tilde{r}^2}\right\}. \tag{6.99}$$

Clearly, this expression will, like (6.88), be equivalent to some multiple of $\delta(\boldsymbol{r})$ in the limit $w \to 0$. Naïvely, one might assume that the integration over the angular coördinates of $(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{a})(\boldsymbol{n}_{\tilde{r}}\cdot\boldsymbol{b})$ would yield $(\boldsymbol{a}\cdot\boldsymbol{b})/3$, and hence leave us with the overall result $-2\delta(\boldsymbol{r})/3$ for (6.99). However, this ignores the fact that $\boldsymbol{n}_{\tilde{r}}$ itself involves a subtle mix of $\boldsymbol{r}$ and $\tilde{\boldsymbol{r}}$. To obtain the *correct* result, replace $\boldsymbol{n}_{\tilde{r}}$ in favour of $\boldsymbol{n}$, before integrating over $\boldsymbol{r}$:

$$\boldsymbol{n}_{\tilde{r}} \equiv \frac{r}{\tilde{r}}\boldsymbol{n};$$

(6.99) then gives

$$\left\{(\boldsymbol{a}\cdot\boldsymbol{b}) - 5(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{b})\frac{r^2}{\tilde{r}^2}\right\}\frac{3}{4\pi\tilde{r}^3}\left\{1 - \frac{r^2}{\tilde{r}^2}\right\}.$$

We *may* now trivially integrate over the angular coördinates $\boldsymbol{n}$ of $\boldsymbol{r}$—since $\boldsymbol{n}$ involves $\boldsymbol{r}$ only, having no reference to the appended coördinate w—giving $(\boldsymbol{a}\cdot\boldsymbol{b})/3$. We thus complete the integration over r:

$$(\boldsymbol{a}\cdot\boldsymbol{b})\int_0^\infty 4\pi r^2\,dr\,\frac{3}{4\pi\tilde{r}^3}\left\{1 - \frac{r^2}{\tilde{r}^2}\right\} - \frac{5}{3}(\boldsymbol{a}\cdot\boldsymbol{b})\int_0^\infty 4\pi r^2\,dr\,\frac{3r^2}{4\pi\tilde{r}^5}\left\{1 - \frac{r^2}{\tilde{r}^2}\right\};$$

again employing the change of variable (6.87), we find

$$3(\boldsymbol{a}\cdot\boldsymbol{b})\int_0^1 du\,u^2 - 5(\boldsymbol{a}\cdot\boldsymbol{b})\int_0^1 du\,u^4 = 0.$$

Thus, we find that the divergence (6.99) in fact *vanishes*:

$$\nabla\cdot\frac{(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{n} - 3(\boldsymbol{n}\cdot\boldsymbol{a})(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{n}}{4\pi r^2} = 0, \tag{6.100}$$

where again we drop the tildes in this final expression.

We have thus shown that there are no further delta-function contributions to the divergences of the point field expressions. The vanishing of the divergences of the regular function terms is demonstrated by the computer algebra program RADREACT of Section G.6.

6.7 Inverse-cube integrals

In the considerations of Section 6.4, we generally assumed that the angular integration over $\boldsymbol{n}_d$, and radial integration over r_d, may be computed separately. We noted there, however, that we must take special care when it comes to *inverse-cube* integrals that we shall have need to perform, *viz.*, those of the form

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\le\varepsilon}d^3r\int_{r'\le\varepsilon}d^3r'\,r_d^{-3}f(\boldsymbol{n}_d), \tag{6.101}$$

where $f(\boldsymbol{n}_d)$ is an even function of $\boldsymbol{n}_d$. The reason is that, by (6.27), the radial integral η_3 is *infinite*—despite our attempts to avoid infinities by expanding the body into a finite sphere of radius ε. (We could have avoided this problem from the outset by having a *varying* source density throughout the body—in particular, one that vanished at least linearly at the centre,—but that would introduce significant complications into many other computations of this chapter; it is far simpler to live with the relatively few yet subtle considerations of the present section.)

Naturally, we would automatically be in dire straits if there were any *finite* terms of order r_d^{-3} remaining in our final equations of motion after angular integration over $\boldsymbol{n}_d$, since then we would be faced with a hopelessly infinite contribution to our equations of motion. In practice, however, this is not the case: we shall find that the angular $\boldsymbol{n}_d$ dependencies of the r_d^{-3} terms are always such that they *cancel* on integration.

The problem, of course, is that if the angular integration yields zero, yet the radial integration yields infinity, the product of the two could equal anything:

$$0 \cdot \infty = \text{anything};$$

in particular, we would be in serious error if we were to assume that their product vanishes.

To compute integrals such as (6.101) correctly, we must instead employ again the $\tilde{\boldsymbol{r}}$-space of Section 6.6—or, more precisely, $\tilde{\boldsymbol{r}}_d$-space, since we require it only for the $\boldsymbol{r}_d$-space integrals; by keeping the appended coördinate w *finite*, until the end of the radiation reaction computations altogether, we will find that the infinities can be contained, and that they do, in fact, cancel rigorously, leaving purely finite results.

6.7.1 The bare inverse-cube integral

We first consider the inverse-cube integral with $f(\boldsymbol{n}_d) = 1$:

$$\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-2} \int_{r \leq \varepsilon} d^3 r \int_{r' \leq \varepsilon} d^3 r' \, \tilde{r}_d^{-3}.$$

The $\boldsymbol{r}_s$-integral is trivial, as before; we are left with

$$\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-1} \int_0^{2\varepsilon} 4\pi r_d^2 \, dr_d \left\{ 1 - \frac{3}{2} \left(\frac{r_d}{2\varepsilon} \right) + \frac{1}{2} \left(\frac{r_d}{2\varepsilon} \right)^3 \right\} \frac{1}{(r_d^2 + w_d^2)^{3/2}}. \tag{6.102}$$

Now, to avoid wasted effort, we shall, from the outset, keep in mind that, ultimately, we shall be taking the limit $w_d \to 0$; we shall, for this reason, retain the quantity w_d only so far as required to regularise the expressions involved. Thus, the integrals of the second and third terms of (6.102)—which are perfectly well-behaved at both limits of integration—can be performed with $w_d = 0$ immediately:

$$\begin{aligned}
&\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-1} \int_0^{2\varepsilon} 4\pi r_d^2 \, dr_d \left\{ -\frac{3}{2} \left(\frac{r_d}{2\varepsilon} \right) + \frac{1}{2} \left(\frac{r_d}{2\varepsilon} \right)^3 \right\} \frac{1}{(r_d^2 + w_d^2)^{3/2}} \\
&= -\frac{9\eta_0}{4\varepsilon^4} \int_0^{2\varepsilon} dr_d + \frac{3\eta_0}{16\varepsilon^6} \int_0^{2\varepsilon} dr_d \, r_d^2 + \mathrm{O}(w_d) \\
&= -\frac{9\eta_0}{2\varepsilon^3} + \frac{\eta_0}{2\varepsilon^3} + \mathrm{O}(w_d) \\
&= -\frac{4\eta_0}{\varepsilon^3} + \mathrm{O}(w_d).
\end{aligned} \tag{6.103}$$

The integral of the first term in (6.102), however, must be computed with w_d finite: to compute

$$\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-1} \int_0^{2\varepsilon} 4\pi r_d^2 \, dr_d \frac{1}{(r_d^2 + w_d^2)^{3/2}}$$

we must employ the change of variable (6.87) (with r and $\tilde{r}$ replaced by r_d and $\tilde{r}_d$, of course); noting further that

$$r_d^2 = \frac{w_d^2 u^2}{1 - u^2},$$

we find

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-1}\int_0^{2\varepsilon}4\pi r_d^2\,dr_d\frac{1}{(r_d^2+w_d^2)^{3/2}} \;=\; \frac{3\eta_0}{\varepsilon^3}\int_0^{2\varepsilon/(4\varepsilon^2+w_d^2)^{1/2}}du\,\frac{u^2}{1-u^2}. \quad (6.104)$$

Finally, by noting that

$$\int du\,\frac{u^2}{1-u^2}=\ln\left(\frac{1+u}{1-u}\right)^{1/2}-u$$

(which may be verified by direct differentiation), we find

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-1}\int_0^{2\varepsilon}4\pi r_d^2\,dr_d\,\tilde{r}_d^{-3}=\frac{3\eta_0}{\varepsilon^3}\ln\frac{4\varepsilon}{w_d}-\frac{3\eta_0}{\varepsilon^3}+\mathrm{O}(w_d). \quad (6.105)$$

Coupling the result (6.105) with that of (6.103), we thus find that

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\leq\varepsilon}d^3r\int_{r'\leq\varepsilon}d^3r'\,\tilde{r}_d^{-3}=\frac{3\eta_0}{\varepsilon^3}\ln\frac{4\varepsilon}{w_d}-\frac{7\eta_0}{\varepsilon^3}+\mathrm{O}(w_d). \quad (6.106)$$

Clearly, the first term in (6.106) encapsulates the (logarithmic) divergence of the integral in the limit $w_d\to 0$; the second term, on the other hand, represents a finite contribution. To write this result in a somewhat more shorthand notation, we define two new integral constants, η_3' and η_3'' (the unit coëfficients here are arbitarily chosen):

$$\eta_3' \equiv \frac{\eta_0}{\varepsilon^3}, \quad (6.107)$$

$$\eta_3'' \equiv \eta_3'\ln\frac{4\varepsilon}{w_d}; \quad (6.108)$$

we then have

$$\eta_0\left(\frac{4}{3}\pi\varepsilon^3\right)^{-2}\int_{r\leq\varepsilon}d^3r\int_{r'\leq\varepsilon}d^3r'\,\tilde{r}_d^{-3}=-7\eta_3'+3\eta_3'', \quad (6.109)$$

where we take it as understood that there are terms of order w_d present (that will of course have no bearing on the final results). It is the presence of η_3'', of course, that renders the unappended integral η_3 infinite.

6.7.2 Inverse-cube integral with two normals

We now turn to the integral

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\le\varepsilon} d^3r \int_{r'\le\varepsilon} d^3r'\, \tilde{r}_d^{-3} n_{\tilde{r}_d i} n_{\tilde{r}_d j}. \tag{6.110}$$

As we found earlier, we must be careful to replace $\boldsymbol{n}_{\tilde{r}_d}$ by its equivalent expression in terms of $\boldsymbol{n}_d$:

$$\boldsymbol{n}_{\tilde{r}_d} \equiv \frac{r_d}{\tilde{r}_d} \boldsymbol{n}_d.$$

The integral over $\boldsymbol{n}_d$ then proceeds as usual, leaving the result $\delta_{ij}/3$. The integral over r_d is now the same as it was for (6.102), except that the integrand is multiplied by the factor $r_d^2/\tilde{r}_d^2$. This does not, of course, have any effect on the integrals of the second and third terms (in the limit $w_d \to 0$): the result is the same as before:

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-1} \int_0^{2\varepsilon} 4\pi r_d^2\, dr_d \left\{ -\frac{3}{2}\left(\frac{r_d}{2\varepsilon}\right) + \frac{1}{2}\left(\frac{r_d}{2\varepsilon}\right)^3 \right\} \frac{r_d^2}{(r_d^2 + w_d^2)^{5/2}} = -\frac{4\eta_0}{\varepsilon^3} + \mathrm{O}(w_d).$$

On the other hand, the integral of the first term is modified by the appearance of the extra factor $r_d^2/\tilde{r}_d^2$: we have an extra factor of u^2 over that present in (6.104):

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-1} \int_0^{2\varepsilon} 4\pi r_d^2\, dr_d \frac{r_d^2}{(r_d^2 + w_d^2)^{5/2}} = \frac{3\eta_0}{\varepsilon^3} \int_0^{2\varepsilon/(4\varepsilon^2 + w_d^2)^{1/2}} du\, \frac{u^4}{1-u^2}.$$

We now note the convenient identity

$$\frac{u^4}{1-u^2} \equiv \frac{u^2}{1-u^2} - u^2; \tag{6.111}$$

in other words, the multiplication by u^2 is equivalent to simply an extra added term $-u^2$; we thus immediately find

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\le\varepsilon} d^3r \int_{r'\le\varepsilon} d^3r'\, r_d^2 \tilde{r}_d^{-5} = -8\eta_3' + 3\eta_3''. \tag{6.112}$$

For the complete integral (6.110), we therefore have

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r' \, \tilde{r}_d^{-3} n_{\tilde{r}_d i} n_{\tilde{r}_d j} = \frac{\delta_{ij}}{3} \left\{ -8\eta_3' + 3\eta_3'' \right\}. \tag{6.113}$$

We can now see why the apparent vanishing of an $\boldsymbol{n}_d$ integration cannot be trusted, for the inverse-cube fields; for example, from (6.109) and (6.113), we have, for the integral of the dipolar field expression over the interior of the sphere,

$$\begin{aligned} \left(\frac{4}{3}\pi\varepsilon^3\right)^{-1} \int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r' \, \frac{3(\boldsymbol{n}_d \cdot \boldsymbol{a})\boldsymbol{n}_d - \boldsymbol{a}}{4\pi r^3} &= -\eta_3' \boldsymbol{a} \left(\frac{4}{3}\pi\varepsilon^3\right) \\ &= -\frac{1}{3}\boldsymbol{a}, \end{aligned} \tag{6.114}$$

the *correct* result (5.86) of Chapter 5, *despite* the fact that the $\boldsymbol{n}_d$ integral would appear to otherwise vanish. Note that, as advertised, the divergent quantity η_3'' has disappeared from this final result (which will, incidentally, be a contributor to our final equations of motion).

6.7.3 Inverse-cube integral with four normals

Finally, we consider the inverse-cube integral involving four factors of $\boldsymbol{n}_{\tilde{r}_d}$:

$$\eta_0 \left(\frac{4}{3}\pi\varepsilon^3\right)^{-2} \int_{r\leq\varepsilon} d^3r \int_{r'\leq\varepsilon} d^3r' \, \tilde{r}_d^{-3} n_{\tilde{r}_d i} n_{\tilde{r}_d j} n_{\tilde{r}_d k} n_{\tilde{r}_d l}.$$

Clearly, our analysis above for the two-factor case carries through identically, except that the $\boldsymbol{n}_d$-integration yields, of course,

$$\frac{1}{15} \left\{ \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \right\},$$

and that there is now yet another extra factor of u^2 in our integrand. By noting, from (6.111), that

$$\frac{u^6}{1-u^2} \equiv \frac{u^4}{1-u^2} - u^4 \equiv \frac{u^2}{1-u^2} - u^2 - u^4,$$

we immediately find that the r_d-integral's contribution is the same as before, with the addition this time of a term $-u^4$ in the integrand; thus,

$$\eta_0 \left(\frac{4}{3} \pi \varepsilon^3 \right)^{-2} \int_{r \leq \varepsilon} d^3 r \int_{r' \leq \varepsilon} d^3 r' \, \tilde{r}_d^{-3} n_{\tilde{r}_d i} n_{\tilde{r}_d j}$$
$$= \frac{1}{15} \Big\{ \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \Big\} \left\{ -\frac{43}{5} \eta_3' + 3\eta_3'' \right\}. \quad (6.115)$$

6.8 Computation of the self-interactions

We can now use the expressions of the previous sections, as well as the power, force and torque equations of motion of Chapter 4, to obtain the radiation reaction equations of motion themselves.

6.8.1 Non-radiative equations of motion

The equations of motion in Chapter 4, ignoring radiation reaction, were obtained for an arbitrary velocity of the particle. Since we have, in this chapter, placed the "receiving" body *at rest*, we need only consider these equations for the simplified situation $\boldsymbol{v} = \boldsymbol{0}$. For a point charge, electric dipole and magnetic dipole, we then find

$$\begin{aligned}
P_q &= 0, \\
P_d &= d\dot{\boldsymbol{\sigma}} \cdot \boldsymbol{E}, \\
P_\mu &= \mu\dot{\boldsymbol{\sigma}} \cdot \boldsymbol{B}, \\
\boldsymbol{F}_q &= q\boldsymbol{E}, \\
\boldsymbol{F}_d &= d(\boldsymbol{\sigma} \cdot \nabla)\boldsymbol{E} + d(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{E} + d\dot{\boldsymbol{\sigma}} \times \boldsymbol{B}, \\
\boldsymbol{F}_\mu &= \mu(\boldsymbol{\sigma} \cdot \nabla)\boldsymbol{B} + \mu(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{B} - \mu\dot{\boldsymbol{\sigma}} \times \boldsymbol{E} + \mu\boldsymbol{\sigma} \times \boldsymbol{J}, \\
\boldsymbol{N}_q &= \boldsymbol{0}, \\
\boldsymbol{N}_d &= d\boldsymbol{\sigma} \times \boldsymbol{E}, \\
\boldsymbol{N}_\mu &= \mu\boldsymbol{\sigma} \times \boldsymbol{B}. \qquad (6.116)
\end{aligned}$$

6.8.2 Gradient forces

To compute the *gradient forces* for the electric and magnetic dipole moments, in (6.116), we use the chain rule: from (6.11), we have

$$\nabla_r \equiv \nabla_{r_d} + \nabla_{r_s}. \tag{6.117}$$

This identity is used by the program RADREACT to calculate the gradients of the retarded fields, listed in Section G.6.22.

6.8.3 Removal of redundant quantities

The self-interactions involving the electric or magnetic dipole moment as both the sending *and* receiving constituent involve the spin vector $\boldsymbol{\sigma}$, or its derivatives, *twice*. We must therefore ensure that we remove redundant or vanishing quantities that may arise through the dot-producting of these quantities together. In particular, we start with the definition

$$\boldsymbol{\sigma}^2 = 1. \tag{6.118}$$

Differentiating (6.118) successively with respect to lab-time, we have

$$\begin{aligned}
(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\sigma}}) &= 0,\\
(\boldsymbol{\sigma}\cdot\ddot{\boldsymbol{\sigma}}) + \dot{\boldsymbol{\sigma}}^2 &= 0,\\
(\boldsymbol{\sigma}\cdot\dddot{\boldsymbol{\sigma}}) + 3(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}}) &= 0,\\
(\boldsymbol{\sigma}\cdot\ddddot{\boldsymbol{\sigma}}) + 3\ddot{\boldsymbol{\sigma}}^2 + 4(\dot{\boldsymbol{\sigma}}\cdot\dddot{\boldsymbol{\sigma}}) &= 0
\end{aligned} \tag{6.119}$$

There arises a choice here of which dot-product we should replace, for the last three identities. We make the choice that *the first term listed* in each of identities (6.119) is to be replaced using that identity. This choice is of course arbitrary, but it will ensure that higher orders of differentiation are always removed in favour of lower orders where possible.

6.8.4 Duality symmetry considerations

By considering the duality of the fields generated by an electric and a magnetic dipole moment, and the force expressions for these moments,—*excepting* the extra Maxwell magnetic field contribution, and the contact force, for a magnetic dipole,—it is clear that, apart from these exceptions, the self-interactions between d and μ will cancel. (This, of course, relies on the assumption that the two moments are parallel, as is the case for spin-half particles.)

The extra magnetic dipole moment contributions are considered in Sections 6.8.7 and 6.8.8.

6.8.5 Constituent spin derivative

Clearly, at $t = 0$, the time rate of change of the spin of the constituent $\boldsymbol{r}$, namely, $\dot{\boldsymbol{\sigma}}_r(0)$, will be related to that of the body as a whole by means of the chain rule:

$$d_{t_r}\boldsymbol{\sigma}_r|_{t=0} \equiv (d_{t_r}\tau)(d_\tau\boldsymbol{\sigma}_r)|_{t=0} \equiv (d_{\tau_r}\tau)(d_\tau\boldsymbol{\sigma})|_{t=0} \equiv d_{\tau_r}\tau|_{t=0}\dot{\boldsymbol{\sigma}}.$$

But $d_{\tau_r}\tau$ is just the reciprocal of the accelerative redshift factor $\lambda(\boldsymbol{r}) \equiv d_\tau\tau_r$; thus,

$$\dot{\boldsymbol{\sigma}}_r(0) = \frac{1}{\lambda(\boldsymbol{r})}\,\dot{\boldsymbol{\sigma}}. \tag{6.120}$$

(This may also be obtained, as an expansion in t, from the expression (3.24); but the result (6.120) is of course exact.)

In practical terms, the result (6.120) is implemented by simply replacing the factor $\dot{\boldsymbol{\sigma}}$ that appears in each constituent force law of (6.116) by $\dot{\boldsymbol{\sigma}}$ of the body as a whole; the factor of $1/\lambda(\boldsymbol{r})$ "cancels out" the factor of $\lambda(\boldsymbol{r})$ inserted to compute the correct power, force and torque on the body as a whole.

(This latter phenomenon can be understood in simple terms, by recognising that the factors of λ appearing in the power, force and torque expressions were introduced because these quantities are *themselves* derivatives of mechanical quantities, namely, p^0, $\boldsymbol{p}$ and $\boldsymbol{s}$ respectively. Thus, the effects of the "presence of an overdot" on either side of the equation cancel.)

6.8.6 Intrinsic and moment-arm torques

One may at first think that, to compute the total torque on the body as a whole, one would simply need to sum the torques on all of its constituents, weighted by the accelerative redshift factor $\lambda(\boldsymbol{r})$.

However, this is, in fact, only *part* of the answer. We must also recall that our spherical body has a finite extent; thus, we need to take into account the "moment-arm" torque on the body as a whole, *viz.*, the torque due to the *force* on each constituent, pre-crossed by the radius vector $\boldsymbol{r}$ to this constituent. For disambiguation, we shall refer to this "moment-arm" contribution to the torque by the general symbol $\boldsymbol{N}^F$, and the aforementioned "intrinsic" contribution to the torque by the general symbol $\boldsymbol{N}^N$.

We may alternatively view the "intrinsic" and "moment-arm" parts of the torque equation as the time rates of change of the *spin* and *orbital* mechanical angular momenta of the constituent respectively. Of course, from the considerations of Chapter 3, we know that the body "reädjusts" itself so that the orbital mechanical angular momentum remains zero (*i.e.*, the constituents do not actually begin to rotate); but all this means is that the "internal rigidity mechanism" of the body shares the mechanical angular momentum that *would* have been imparted to the constituent in question among the spins of all of the constituents of the body.

6.8.7 Magnetic Maxwell field contribution

We now consider the contributions to the radiation reaction equations of motion arising from the extra magnetic field at the position of a magnetic dipole, over and above that of the dual of the electric dipole field, necessary in order that Maxwell's equations are satisfied; we dubbed this extra field the *extra magnetic Maxwell field* in Chapter 5 (for want of a better name).

We found, in that chapter, that, for a *point* magnetic dipole, the extra Maxwell field is given, at the instant the dipole is at rest, by

$$\boldsymbol{B}_M = \mu\boldsymbol{\sigma}\,\delta(\boldsymbol{r}), \tag{6.121}$$

regardless of whether the body is accelerated or not. (*See* equations (5.92) and (5.106) of Chapter 5.)

For a spherical body of radius ε, one can either use (6.121) directly—by noting that the delta-function appearing there is identical to the delta-function dipole moment density of a point particle,—or simply examine the expressions for such an extended body found in Section 5.5.3; either way, one finds

$$\boldsymbol{B}_M = \begin{cases} \dfrac{3\mu}{4\pi\varepsilon^3}\,\boldsymbol{\sigma}, & r < \varepsilon, \\ \\ \boldsymbol{0}, & r > \varepsilon. \end{cases} \tag{6.122}$$

Now, the Maxwell field (6.122) will not lead to any contribution to the *power* equation of motion for the receiving magnetic dipole constituents, by virtue of the identity

$$(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{\sigma}}) = 0.$$

It also, trivially, does not contribute to the receiving *electric* dipole constituents' power equation, since the latter depends on the *electric* field only.

The Maxwell field does, however, contribute to the *force* and *torque* equations of motion. The "intrinsic" contribution to the latter vanishes, trivially in the electric dipole case, and by virtue of the identity $\boldsymbol{\sigma}\times\boldsymbol{\sigma} \equiv \boldsymbol{0}$ for the magnetic dipole case.

We thus concentrate on the force equations of motion. The Maxwell field does not affect the receiving electric charges, as their force depends only on the electric field. There is, however, a force on the *electric dipole moment*: we have

$$\begin{aligned}\boldsymbol{F}_{\mu d}^{(M)} &= \frac{3d}{4\pi\varepsilon^3}\int_{r\leq\varepsilon} d^3r\,\frac{3\mu}{4\pi\varepsilon^3}\dot{\boldsymbol{\sigma}}\times\boldsymbol{\sigma}\\ &= -\frac{3d\mu}{4\pi\varepsilon^3}\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}}\\ &\equiv -3d\mu\eta_3'. \end{aligned}\tag{6.123}$$

For the corresponding "moment-arm" *torque* contribution, we note that the above expressions are exact (there is no factor of λ since the electric dipole force depends on $\dot{\boldsymbol{\sigma}}$), and they are even in $\boldsymbol{r}$; hence, when we cross $\boldsymbol{r}$ into this expression, the resulting odd integrand vanishes upon integration over the three-sphere of $\boldsymbol{r}$. There is therefore no torque on the electric dipole due to the Maxwell field.

For force on the magnetic dipole moment *itself*, due to its extra Maxwell field, we must be a little more careful. Since the Maxwell field is of a contant value *inside* the sphere, and zero *outside* the sphere, we find the only non-zero gradient of the field is of a *Dirac delta-function* form, around the surface of the sphere. Now, the subtlety arises because the magnetic dipole moment density *itself* has a sharp transition around this surface. However, this step function in moment density is of course the *same* step function that leads to the Dirac delta function in the gradient of the Maxwell field itself (as can be seen by noting its source, equation (6.121)). We are therefore led to consider integrals over (generalised) functions of the form

$$\int dt\,\delta(t)\vartheta(t)f(t) \equiv \frac{1}{2}f(0),\tag{6.124}$$

where the right-hand side of this identity may be trivially verified by integrating by parts.

Now, the gradient of the Maxwell field (6.122) is given, from first principles, by

$$(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B}_M = -\frac{3\mu}{4\pi\varepsilon^3}(\boldsymbol{\sigma}\cdot\boldsymbol{n})\boldsymbol{\sigma}\,\delta(r-\varepsilon).\tag{6.125}$$

This asserted result of the author's may be verified in four steps. Firstly, we note that the *magnitude* of (6.125), $3\mu/4\pi\varepsilon^3$, is simply given by the magnitude of the Maxwell field, since the step is from this value to zero. Secondly, the factor $(\boldsymbol{\sigma}\cdot\boldsymbol{n})$ in (6.125) is unity when $\boldsymbol{n}$ is in the direction of $\boldsymbol{\sigma}$, negative unity when $\boldsymbol{n}$ is antiparallel to $\boldsymbol{\sigma}$, and zero when $\boldsymbol{n}$ is perpendicular to $\boldsymbol{\sigma}$; this factor embodies the fact that the step occurs as we move radially outwards. Thirdly, the minus sign of (6.125) reflects the fact that the step is from B_M *down to* the value zero. Fourthly, the vector $\boldsymbol{\sigma}$ is the direction of the Maxwell field itself.

To compute the self-force on the body as a whole, we multiply (6.125) by the redshift factor $\lambda(\boldsymbol{r})$, the step function $\vartheta(\varepsilon - r)$ that cuts off the receiving moment density at the surface, and integrate over *all* space; since (6.125) is odd, we must select the odd part $r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})$ of $\lambda(\boldsymbol{r})$:

$$\boldsymbol{F}_{\mu\mu}^{(M)} = -\left(\frac{\mu}{4\pi\varepsilon^3}\right)^2 \int d^3r\, r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{\sigma}\cdot\boldsymbol{n})\boldsymbol{\sigma}\,\delta(r-\varepsilon)\,\vartheta(\varepsilon - r).$$

In performing this integral, we note that the angular integration will yield a factor of 1/3, and the presence of the delta and step functions a factor of 1/2; the remaining factors in the integrand contribute $4\pi\varepsilon^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$, and hence

$$\begin{aligned}\boldsymbol{F}_{\mu\mu}^{(M)} &= -\frac{1}{3}\cdot\frac{1}{2}\cdot 4\pi\varepsilon^3\left(\frac{\mu}{4\pi\varepsilon^3}\right)^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\ &= -\frac{3}{2}\mu^2\eta_3'(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}.\end{aligned}$$

For the "moment-arm" torque following from this same force, we must of course now choose the even part (unity) of $\lambda(\boldsymbol{r})$, and then cross the vector $\boldsymbol{r}$ into the result. However, we shall then be integrating an expression involving the factor $(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\boldsymbol{\sigma}$, which, after angular integration, yields $\boldsymbol{\sigma}\times\boldsymbol{\sigma}$, and hence vanishes.

6.8.8 Magnetic contact force contribution

We now consider the contribution to the radiation reaction force equation of motion of the *contact force* on the magnetic dipole moments of the constituents.

Since this force depends on the current $\boldsymbol{J}$, it may be wondered why we are considering it at all: where would this current come from? The answer, of course, is found in the considerations of Chapter 5: the magnetic dipole moment itself has a *current sheet* surrounding it, which is actually responsible for generating the moment. The current density $\boldsymbol{J}(\boldsymbol{r})$ is given by

$$\boldsymbol{J}(\boldsymbol{r}) = \frac{3\mu}{4\pi\varepsilon^3}\,\boldsymbol{\sigma}\times\boldsymbol{n}\,\delta(r-\varepsilon). \tag{6.126}$$

Again, we can verify (6.126) in four steps. Firstly, we note that the *direction* of the cross-product is such that the sheet of current is circulating in the correct sense to produce a magnetic field in the direction of $\boldsymbol{\sigma}$. Secondly, we note that the *magnitude* of the cross-product (namely, of the form $\sin\theta$, where θ is the "latitude" on the sphere, if $\boldsymbol{\sigma}$ points to the "North Pole") is maximum perpendicular to $\boldsymbol{\sigma}$, and vanishes in the directions parallel and antiparallel to $\boldsymbol{\sigma}$, in agreement with the analysis of Chapter 5. Thirdly, the coëfficient of (6.126) is the correct stepping of the magnetic field from its internal value to its external value; this may be verified quickly by noting that, *before* we added the Maxwell field, the field matched *smoothly* around the "Equator"; the step in the field around this circular boundary is therefore just that of the Maxwell field. Fourthly, the delta function reflects the fact that it is a sheet current.

Now, the contact force contribution of each constituent will be

$$\lambda(\boldsymbol{r})\,\boldsymbol{\sigma}\times\boldsymbol{J}(\boldsymbol{r}) = \frac{3\mu}{4\pi\varepsilon^3}\lambda(\boldsymbol{r})\,\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\boldsymbol{n})\,\delta(r-\varepsilon). \tag{6.127}$$

Because (6.126) is *odd* in $\boldsymbol{r}$, we need to use the odd part $r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})$ of $\lambda(\boldsymbol{r})$ to obtain a non-vanishing integral over all space. We then have

$$\boldsymbol{F}^{(C)}_{\mu\mu} = \left(\frac{\mu}{4\pi\varepsilon^3}\right)^2 \int d^3r\, r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\Big\{(\boldsymbol{\sigma}\cdot\boldsymbol{n})\boldsymbol{\sigma} - \boldsymbol{n}\Big\}\,\delta(r-\varepsilon)\,\vartheta(\varepsilon - r).$$

where the superscript $^{(C)}$ denotes that it is the contact force contribution. (We count this contribution together with those of superscript $^{(M)}$ in the program RADREACT.) The integral of the first term is identical to that of the previous section, except reversed in sign; the integral of the second term has the same numerical factor (reversed back in sign again) because it also contains two factors of $\boldsymbol{n}$. Hence,

$$\boldsymbol{F}^{(C)}_{\mu\mu} = \frac{3}{2}\mu^2\eta'_3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{3}{2}\mu^2\eta'_3\dot{\boldsymbol{v}}.$$

The corresponding "moment arm" contribution to the torque vanishes, in the case of the first term of the integral, by the same arguments as the previous section; and, in the case of the second term, simply because it involves the cross-product $\boldsymbol{n}\times\boldsymbol{n}$.

6.8.9 Final radiation reaction equations of motion

Finally, we compute the self-interaction expressions themselves. These are, from the above considerations, obtained by means of the relations

$$\begin{aligned}
P^{(n)}_{ab} &= \int_{V_d} d^3r_d \int_{V_s} d^3r_s\, P^{(n)}_{ab}(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}^{(n)}_{ab} &= \int_{V_d} d^3r_d \int_{V_s} d^3r_s\, \boldsymbol{F}^{(n)}_{ab}(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{N}^{(n)}_{ab} &= \boldsymbol{N}^{N(n)}_{ab} + \boldsymbol{N}^{F(n)}_{ab},\\
\boldsymbol{N}^{N(n)}_{ab} &= \int_{V_d} d^3r_d \int_{V_s} d^3r_s\, \boldsymbol{N}^{N(n)}_{ab}(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{N}^{F(n)}_{ab} &= \int_{V_d} d^3r_d \int_{V_s} d^3r_s\, \boldsymbol{r}(\boldsymbol{r}_d,\boldsymbol{r}_s)\times\boldsymbol{F}^{(n)}_{ab}(\boldsymbol{r}_d,\boldsymbol{r}_s),
\end{aligned}$$

where

$$
\begin{aligned}
P_{aq}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= 0,\\
P_{ad}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
P_{a\mu}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{aq}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{E}_n^q(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{ad}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}})\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+\dot{\boldsymbol{\sigma}}\times\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{a\mu}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}})\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)-\dot{\boldsymbol{\sigma}}\times\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)\\
&\quad+\lambda\boldsymbol{\sigma}\times\boldsymbol{J}_n^a,\\
\boldsymbol{N}_{aq}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \boldsymbol{0},\\
\boldsymbol{N}_{ad}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{\sigma}\times\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{N}_{a\mu}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{\sigma}\times\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),
\end{aligned}
$$

where n is the inverse power of R of the retarded fields in question (or M for the Maxwell field of the magnetic dipole), and a and $b = q$, d or μ.

Defining, for convenience, the quantity

$$\tilde{\mu}^2 \equiv d^2 + \mu^2$$

which appears for all dually-symmetric dipole self-interactions, the program RADREACT of Section G.6 finds the following final equations of motion:

$$
\begin{aligned}
P_{\text{self}} &= -\frac{2}{3}qd\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})-\frac{2}{3}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})-\frac{1}{30}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad+\frac{2}{3}qd\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})+\frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{\sigma}}\cdot\dddot{\boldsymbol{\sigma}})+\frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad-\frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}),
\end{aligned}
\tag{6.128}
$$

$$
\begin{aligned}
\boldsymbol{F}_{\text{self}} &= -\frac{3}{2}\mu^2\eta_3'\dot{\boldsymbol{v}}-\frac{1}{2}\tilde{\mu}^2\eta_3'\dot{\boldsymbol{v}}-\frac{1}{2}q^2\eta_1\dot{\boldsymbol{v}}-\frac{2}{3}qd\eta_1\ddot{\boldsymbol{\sigma}}+\frac{4}{15}\tilde{\mu}^2\eta_1\dddot{\boldsymbol{v}}+\frac{2}{3}\tilde{\mu}^2\eta_1\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\\
&\quad-\frac{1}{3}\tilde{\mu}^2\eta_1\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}}+\frac{1}{3}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}}+\frac{1}{15}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}}-\frac{1}{6}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{1}{5}\tilde{\mu}^2\eta_1(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}-\frac{19}{30}\tilde{\mu}^2\eta_1(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&-\frac{2}{15}\tilde{\mu}^2\eta_1(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}-\frac{2}{5}\tilde{\mu}^2\eta_1\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+\frac{2}{3}q^2\eta_0\ddot{\boldsymbol{v}}+\frac{2}{3}qd\eta_0\ddot{\boldsymbol{\sigma}}\\
&-\frac{1}{3}qd\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-qd\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{4}{15}\tilde{\mu}^2\eta_0\dddot{\boldsymbol{v}}-2\tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\\
&+\frac{2}{3}\tilde{\mu}^2\eta_0\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}}-2\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}}-\frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}}\\
&+\frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}}+\frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}}\\
&+\frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{2}{15}\tilde{\mu}^2\eta_0(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+2\tilde{\mu}^2\eta_0(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+2\tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+\frac{5}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+\frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-3d\mu\eta_3'\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}}-\frac{2}{3}q\mu\eta_1\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{2}{3}q\mu\eta_1\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{4}{3}q\mu\eta_0\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{2}{3}q\mu\eta_0\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+2q\mu\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}, \qquad (6.129)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{N}_{\text{self}} &= -\frac{2}{3}q\mu\eta_1\dot{\boldsymbol{\sigma}}\\
&\quad+\frac{1}{2}qd\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{2}\tilde{\mu}^2\eta_1\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}}+\frac{2}{15}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{2}{3}qd\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&\quad+\frac{2}{3}\tilde{\mu}^2\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}}-\frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}. \qquad (6.130)
\end{aligned}
$$

We shall defer, in this thesis, any reëxpression of these final equations of motion in manifestly covariant terms, pending the issues raised in the following discussion. However, the necessary quantities for such a reëxpression are all provided in Section G.4.

6.9 Discussion of the final equations

The author, having opened an arguably large can of worms by attacking all of the problems considered in this thesis, thought he had finally found a lid to this can, by writing the computer algebra programs of Appendix G to complete the radiation reaction computations for him before the termination of his Ph.D. scholarship (indeed, looking now at the expressions in Appendix G, before the end of the *millenium*). He has, however, discovered that this lid is in fact the base of an even *larger* can of worms, arising from the final equations of motion (6.128), (6.129) and (6.130) above. He is informed by those of wiser years that this is actually a general phenomenon of Nature: as far as anyone can tell, it's all worms from here on up.

It would therefore probably be either overly ambitious, or else foolish (or quite possibly both), for the author to attempt a comprehensive discussion of all of the issues that arise from a consideration of equations (6.128), (6.129) and (6.130). Instead, we shall only give here a brief indication of the successes of these equations, their failures, their questionable aspects, their complications, and their intriguing properties. We shall also, in the following section, analyse a simple yet instructive application using just a small subset of these equations, which will bring their successes and limitations into starkest relief.

Let us begin with the unquestionable successes of the equations (6.128), (6.129) and (6.130) (which we shall, for sanity, refer to as simply the "final equations" for the remainder of this section). Firstly, it will be noted that, as promised, the electric charge mass term

$$-\frac{1}{2}q^2\eta_1\dot{\boldsymbol{v}}$$

has the *correct* coëfficient: in Section 5.5.8 of Chapter 5, we found that

$$-\frac{1}{2}q^2\eta_1 \equiv m^q_{\text{e.m.}}.$$

This is in contrast to the incorrect factor of 4/3 in Lorentz's [137] result, equation (6.1). Now, it is instructive to note just *which* aspect of the theoretical framework used in this thesis repaired Lorentz's erroneous result.

If one examines the author's calculations in fine detail, one finds that the inclusion of the fully relativistic *trajectories* of the constituents *does not provide any modification of the Lorentz result at all*. (In fact, the author was completely crestfallen when he originally found this result.) In retrospect, this unsuccessful result was, in fact, *already anticipated*; the reason is subtle, yet instructive: Already, in Heitler's textbook [104], it was noted that "*for high velocities we have to assume a Lorentz contraction*"; but the factor 4/3 was still obtained. Now, when one considers the question of just *what* place the "FitzGerald–Lorentz contraction" has in our current understanding of *accelerated* rigid bodies, one finds that it does, in fact, give the appropriate relativistic correction to the trajectories of the constituents (at least, the leading order non-trivial correction), but it *equates proper-time hypersurfaces with lab-time hypersurfaces*; *i.e.*, it gives a correct trajectorial description, but in terms of Newton's "universal time". Thus, when the author originally included *only* trajectorial corrections, he of course found a result in accord with that given in Heitler.

The crucial ingredient in the correction of the Lorentz method of derivation is, in fact, the *accelerative redshift factor* $\lambda(\boldsymbol{r})$. Indeed, one only needs to simply multiply this factor into Lorentz's original *Galileanly*-rigid computation—without even any FitzGerald–Lorentz contraction—and one finds his 4/3 factor corrected to unity immediately! This feature of the relativistically correct rigid body formalism was, essentially, discussed with great clarity by Pearle [168]: although the derivation therein is based on the Dirac [68] conservation method, the features of the rigid body formalism are the same as those used here; of particular relevance is the informative discussion of the "wedge-shaped hypervolume", which of course represents the effects of the "tilting" of the hypersurfaces, from whence λ derives.

The author has not explicitly seen it pointed out in the literature that the essence of the Lorentz method of derivation is so easily repaired, through simply a recognition of the need for λ; but then again the author has not seen too many uses of the Lorentz method *full stop*: the Dirac method is generally preferred.

The second major success of the final equations is the similarly correct inertial term for the *electric dipole moment*,

$$-\frac{1}{2}d^2\eta_3'\dot{\boldsymbol{v}};$$

again, as shown in Section 5.5.9 of Chapter 5, the field contribution to the mass of the electric dipole is indeed

$$m^d_{\text{e.m.}} = \frac{1}{2}d^2\eta_3'.$$

That this result is not trivial is recognised by considering the various subtleties involved with the inverse-cube integrals of the previous sections, from whence this term derives.

The author will claim, as a third major success, the absence of any "nasty coëfficients" in the final equations. Namely, the coëfficients are all simple fractions; the worst of them is 19/30. (They are not *quite* identical to the set of coëfficients found by Bhabha and Corben, but close enough that the different methods of expression could easily transform one set into the other.) That *this* property is non-trivial is seen just by considering some of the terms in even the *penultimate* expressions of Section G.6: one has, among the honourable entries there listed, terms with coëfficients such as 71/140, 191/210, 127/420 and 331/840. That these coëfficients should all add up to give nice wholesome numbers like 1/3 and 4/15 is an agreeable finding; one would not, arguably, be able to sleep quite as soundly if one were to be told that some fundamental equation of physics were to have the number 331/840 appearing in it.

While we are on the topic of mathematical simplicity, we shall claim a fourth success of the final equations, alluded to in earlier sections: they are "mistake-invariant". Let us explain. When the author first manually performed the electric charge radiation reaction calculations, fully relativistically, and then again when he wrote the program RADREACT to compute the dipole results, he did *not* correctly Thomas-precess the constituent's position $\boldsymbol{r}$ as seen in the lab frame, as now described in Chapter 3; rather, the vector $\boldsymbol{u}(\tau)$ was effectively *fixed*. This subtle error of philosophy was not detected until the program was fully completed and debugged. When the calculations were suitably repaired, it was found that the changes carried through to *three orders* of terms in the trajectories of the constituents. This had consequential changes on about *a third of all terms appearing in Section G.6!* These changes carried right through (literally) ninety-nine pages of expressions. Then, in the *final equations only*, all of these changes cancelled out completely! The author was not prepared for such an outcome.

The author believes this phenomenon may be explained as follows. By essentially "rotating" the body as it began to move, the trajectory of any *particular* constituent was modified. But, taken as a whole, the relevant properties of the body still seemed to act the same as they would have *without* rotation, perhaps because as one constituent rotated out of position, another came to fill its place. This clearly is not correct physically, but it was probably correct *mathematically* by virtue of the other underlying assumptions of the author based on the assumption that the body *did* remain non-rotating. Regardless, the results magically returned to their original form on the hundredth page.

That this behaviour casts a favourable outlook on the computer algebra computations follows from the fact that, regardless of the exact reason *why* the results were mathematically invariant, the fact that the computer algebra program *did* in fact return us to the same answers, after ninety-nine pages of differences, shows that it must be computing at least *something* robustly; and from the relative simplicity of the results, one would suspect that this something might well be physical reality. Of course, this is not to say that there are not other completely different contributions to the equations of motion, overlooked by the author, that are totally independent of the quantities computed in this thesis, that would not "mix" with the terms herein under the "mistake transformation" anyway; but at least the relative independence of such possibly overlooked contributions has been demonstrated.

We finally propose a fifth successful property of *the computations leading to* the final equations, that the author, having lived with the problem for the best part of a year, has a good intuitive understanding of, but which may fail to excite any emotion in the reader: the expressions tend to "automatically" weed out terms that would be difficult to integrate. Let us explain. When one goes through the expressions of Section G.6 term-by-term (say, for simplicity, the electric charge field expressions), one finds that there are terms that, by the normal rules of combinatorics, *should* by all rights be present, but in fact are not. Then one finds that, when one multiplies the expression by λ, or performs some other operation that yields the final interaction equation that is to be integrated, one finds that the "missing" types of term are of such a form that they would give one additional types of nasty "mixed integrals" (involving both $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$) to perform. Now, in the previous sections, we did have need to consider a *few* cases where the inner $\boldsymbol{r}_s$ integral had to be performed explicitly; but, if one considered the problem on purely dimensional grounds, one would expect many more such cases to appear. (Indeed, until the final dipole equations of motion were considered by computer, all required integrals were independent of $\boldsymbol{r}_s$ altogether!) This property of the computations is, as admitted above, somewhat nebulous, but is nevertheless clearly discernible if one contemplates the expressions in the Appendix for timescales on the order of years.

We now turn to two aspects of the final equations that may only be termed "qualified successes". These are the mass renormalisation term for the *magnetic* dipole moment,

$$-2\mu^2\eta_3'\dot{\boldsymbol{v}},$$

and the *spin* renormalisation term for the charged magnetic dipole moment,

$$-\frac{2}{3}q\mu\eta_1\dot{\boldsymbol{\sigma}}.$$

The good news is that they are both there. The bad news is that they are both precisely *twice as large as they are supposed to be*, based on the results found in Chapter 5: we found there that

$$m^{\mu}_{\text{e.m.}} = \mu^2\eta'_3$$

and

$$\boldsymbol{s}^{q\mu}_{\text{e.m.}} = \frac{1}{3}q\mu\eta_1\boldsymbol{\sigma}$$

respectively. Now, the author has checked his algebraic computations of these quantities on numerous occasions, and cannot find any errors of mathematics. Keeping this possibility, of course, always in the back of our minds, let us consider how this result could possibly be explained physically, if it *is* in fact mathematically sound. The author's only hypothesis, at the present time, is that perhaps we are feeling the ramifications of the subtle method of derivation of the dipole equations of motion in Chapter 4. The author does not know. The terms are twice as large as they should be. Suggestions welcome.

We now turn to a property of the final equations that could not quite be termed a success, but rather a lack of a possible failure: there are no terms involving η_2 at all. Again, mathematically speaking, this is non-trivial: some of the expressions that are used to obtain the final results are as high as the minus-*fourth* power in r_d; numerous r_d^{-2} terms appear in the penultimate equations, but they all cancel. That this is a good thing can be seen on two grounds: Firstly, Bhabha and Corben found numerous terms involving ε^{-3} and ε^{-1}, but only a *few* involving ε^{-2}; perhaps these cancel for the assumptions of this thesis. Secondly, and more importantly, there were in fact *no* mechanical self-field quantities computed in Chapter 5 that depended on η_2 (indeed, *nothing* physical in this thesis depends on η_2); if such terms *were* to appear, we would not have anything to blame them on!

Having raised the question of scapegoats, we now turn to those terms in the final equations that are, at present, rather undesirable, but which could of course be vital if we were expecting them: the other infinite terms. On the basis of the Bhabha–Corben analysis, we should expect numerous infinite terms *to* occur, in the point limit: many of their terms also involved ε^{-3} and ε^{-1}. Firstly, there is one infinite term, not dealt with above, that is proportional to η'_3, in the force equation:

$$-3d\mu\eta'_3\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}};$$

this was the obtained explicitly in Section 6.8.7, and inserted by hand into the program. The author has no good physical explanation of this term at present.

Then there are the numerous terms involving qd. Some of these appear in the power equation—which should vanish in the rest frame; others are just proportional to η_1, and do not look like they would vanish for the actual motion of the particle. However, we already knew, from the considerations of Chapter 5, that the charged electric dipole has a *non-zero centre of energy shift*; to counterbalance this, we would need to put a "mass dipole" at the origin. If one considers the problem quantitatively, one finds that the inertial forces imparted by such a "mass dipole" are of just the right form to cancel the bothersome terms in the final equations. We shall, however, leave a complete analysis of this problem to another place.

We finally turn to the remaining infinite terms in the final equations, proportional to $\tilde{\mu}^2\eta_1$, and the remaining finite terms in the self-power, proportional to $\tilde{\mu}^2\eta_0$. There are three possibilities that the author can see for these terms. The first is that perhaps they arise through *higher moments* of the (anisotropic) mechanical energy density of the dipole moments. This is pure speculation. The second is that they may well *vanish for the actual* *motion of the particle*. This has, in fact, been proved by the author to lowest order for the finite terms in the self-power expression; but an analysis of the infinite terms in the *force and torque* equations of motion carries the complication that *higher derivatives* of the kinematical quantities must be calculated; even going to the rest frame does *not* render this trivial, since the differentiations can carry factors of $\boldsymbol{v}$ down to lowest-order, that do not appear in the rest frame before differentiation. In any case, if these infinite terms *were* to be shown to vanish for the actual motion, then we would have the additional problem of principle that, *a priori*, terms of the rigid body equations of motion *of order* ε^{+1}—that we have neglected—could well interact with those of order ε^{-1} (before the point limit is taken, of course), and yield *finite* contributions to the results. It would be expected that this would *not* be likely: if terms vanish for the motion to one approximation, it is probably for a good fundamental reason, and not just a fluke of that order of analysis. But this, too, remains speculation.

Finally, there is the possibility that the assumptions underlying the author's calculations are just plain wrong. If so, no amount of discussion could save them. The author can only hope they are not.

6.10 Sokolov–Ternov and related effects

In this final section, we consider one simple yet testing application of the results of the previous section: the *radiation reaction torque on a charged magnetic dipole*.

This problem was fully treated, quantum mechanically, in the ultrerelativistic limit, by Sokolov and Ternov [198] (first suggested in print by Ternov, Loskutov and Korovina [210]), and has proved to be of immense importance for polarised spin physics in electron storage rings (*see*, *e.g.*, [155]). The case of a neutral particle, in the *non*relativistic limit, was likewise treated, quantum mechanically, by Ternov, Bagrov and Khapaev [212].

Subsequently, Lyuboshitz [141] showed that the Ternov–Bagrov–Khapaev result could be simply and intuitively understood as a radiation reaction spin-flip due to the spontaneous emission of magnetic dipole radiation; but also noted that a generalisation of his heuristic argument to the case of *charged* particles was not trivial. Jackson's comprehensive review [114] examined the Lyuboshitz argument in detail, and explained most clearly why it is essentially completely correct for neutral particles, and *almost*—but not quite—correct for charged particles.

Clearly, a phenomenon that is so manifestly an effect of the reaction of radiation is a suitable test for the author's results. The fact that the quantum analysis involves *spontaneous* radiation warns us that the classical analysis will not be complete; but nevertheless we would, following Schwinger [185, 186], expect the *dimensional* quantities appearing in the results to be completely classical; the spontaneity of the radiation—essentially arising through the use of discrete integers rather than continuous reals for the electromagnetic field—yielding important but dimensionless corrections.

In Section 6.10.1, we briefly review the Sokolov–Ternov effect, followed by the Ternov–Bagrov–Khapaev effect in Section 6.10.2. Lyuboshitz's elementary explanation of the Ternov–Bagrov–Khapaev effect in terms of spin-flip due to spontaneous radiation is presented in Section 6.10.3, and Jackson's comments on both this and the Sokolov–Ternov effect are reviewed in Section 6.10.4. Then, in Section 6.10.5, we consider the question from the point of view of the equations of the previous section, and compare the results with those of Lyuboshitz and Jackson.

6.10.1 The Sokolov–Ternov effect

In a six-paragraph note to *JETP* in 1961 [210], Ternov, Loskutov and Korovina (with acknowledgments to Sokolov) noted that, due to the fact that a Dirac electron moving perpendicular to a uniform magnetic field emits "synchrotron" radiation (*i.e.*, the radiation computed from the Dirac equation—*not* the Liénard–Wiechert radiation of a classical charge) in a highly *asymmetrical* manner [197], the associated spin-flip transition rates, along the axis of the magnetic field, are *not equal*. This was put on a more quantitative basis by Sokolov and Ternov [198], who considered *all* of the relevant spin-flip terms, and showed that this "transverse" polarisation

$$P_t \equiv (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{B}}) \tag{6.131}$$

(*i.e.*, transverse to the velocity and acceleration; *viz.*, in the direction of the magnetic field) builds up, for zero initial polarisation, according to the relation

$$P_t(t) = P_{\mathrm{ST}}\left(1 - e^{-t/\tau_{\mathrm{ST}}}\right), \tag{6.132}$$

where the asymptotic polarisation, P_{ST}, is given by

$$P_{\mathrm{ST}} = \frac{8}{5\sqrt{3}} \sim 92.4\%, \tag{6.133}$$

and the characteristic time, τ_{ST}, by

$$\tau_{\mathrm{ST}} = \left\{ \frac{5\sqrt{3}}{8} \frac{e^2\hbar\gamma^5}{4\pi m^2 R^3} \right\}^{-1} \equiv \left\{ \frac{5\sqrt{3}}{8} \frac{\lambda\!\!\!^{-}_C r_c}{R^3} \right\}^{-1}, \tag{6.134}$$

where R is the radius of curvature of the trajectory, $\lambda\!\!\!^{-}_C \equiv \hbar/m$ is the reduced Compton wavelength, $r_c \equiv e^2/4\pi m$ is the classical electron radius, and we are still using the notational conventions and units of Appendix A. That the Sokolov–Ternov effect (as it has become known) is, relatively speaking, an extremely small effect is recognised by the presence of the ratios $\lambda\!\!\!^{-}_C/R$ and r_c/R: the Compton wavelength and classical radius of an electron are truly *tiny* compared to the radii of typical terrestrial storage rings. The strong energy dependence (γ^5) of the polarisation rate ameliorates the situation somewhat; the resulting characteristic polarisation times of real rings are, roughly speaking, on the order of *minutes to hours*: extremely long compared to the precession and orbital periods, but nevertheless within the domain of experimental practicality.

The analysis of the Sokolov–Ternov effect was further generalised to the case of arbitrary magnetic field configurations by Baier and Katkov [17]; they found that the spin-flip transition probability per unit time for relativistic electrons or positrons is given, in the general case, by

$$w = \frac{5\sqrt{3}}{16}\frac{e^2\hbar}{m^2}\gamma^5 \left|\dot{\boldsymbol{v}}\right|^3 \left\{ 1 - \frac{2}{9}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{v}})^2 + \frac{8\sqrt{3}}{15}\boldsymbol{\sigma}\cdot\hat{\boldsymbol{v}}\times\hat{\dot{\boldsymbol{v}}} \right\}, \qquad (6.135)$$

where $\hat{\boldsymbol{v}}$ and $\hat{\dot{\boldsymbol{v}}}$ are unit vectors in the direction of $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ respectively. For a circular orbit in a uniform magnetic field, the Baier–Katkov equation (6.135) reduces to the Sokolov–Ternov results (6.132), (6.133) and (6.134).

As an extension of this work, Baier, Katkov and Strakhovenko [18] derived a general equation of motion for the polarisation vector, incorporating the Thomas–Bargmann–Michel–Telegdi and Baier–Katkov equations:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\times\boldsymbol{\Omega}_{\mathrm{TBMT}} - \frac{1}{\tau_{\mathrm{ST}}}\left\{ \boldsymbol{\sigma} - \frac{2}{9}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{v}})\hat{\boldsymbol{v}} + \frac{8}{5\sqrt{3}}\hat{\boldsymbol{v}}\times\hat{\dot{\boldsymbol{v}}} \right\}, \qquad (6.136)$$

where $\boldsymbol{\Omega}_{\mathrm{TBMT}}$ is the Thomas–Bargmann–Michel–Telegdi spin precession frequency vector.

In making any connection with classical physics, we should of course use the Baier–Katkov–Strakhovenko equation, (6.136), as the general expression encompassing the Sokolov–Ternov effect, for electrons and positrons in arbitrary relativistic motion. However, for the purposes of this simplified analysis, we shall consider only the simplified Sokolov–Ternov configuration.

We shall also refrain, here, from discussing the many exciting advances being made in polarised beam physics in high energy storage rings, but will instead refer the interested reader to some entry points in the literature: [155, 21, 22, 23, 15, 123].

6.10.2 The Ternov–Bagrov–Khapaev effect

Closely related to the Sokolov–Ternov effect is the question of the radiation emitted by, and subsequent radiation reaction on, a neutron traversing a magnetic field. Clearly, in a unified and general treatment, this physical situation would be included as a subset of the Sokolov–Ternov phenomenon, since one would then only need to set the value of the electric charge to zero to have a magnetic dipole moment alone.

However, in practice, the neutron case needs to be treated differently. The reasons for this are as twofold. Firstly, to analyse a neutron, one must of course include the effects of an *anomalous* magnetic moment; this contribution is relatively negligible for an electron, and hence an analysis of the minimally-coupled Dirac equation is quite sufficient for the latter situation. (The arbitrary-g case has, however, been treated by Derbenev and Kondratenko [67], and in the review by Jackson [114] to be discussed shortly; but the analysis is quite complicated.) Secondly, the electron results are generally applied to *high energy* storage rings; given the relatively small mass of the electron, practically *any* modern energy is ultra-relativistic, and hence this limit is generally employed theoretically, as being experimentally applicable, to an extremely good approximation. But, for obvious reasons, we do not have analogous high energy *neutron* storage rings; rather, we generally find neutrons travelling in straight lines—at least, we don't deflect them, terrestrially, using electromagnetic fields. Thus, we are generally more interested in the *nonrelativistic* limit of neutron motion (which may, of course, be Lorentz-boosted to any arbitrary energy if we so wish). Clearly, one cannot apply ultra-relativistic electron results to nonrelativistic neutrons.

Thus, Ternov, Bagrov and Khapaev [212] considered from first principles the case of a nonrelativistic neutron in a magnetic field. The obtained expressions for the radiation fields, from which they calculated, *à la* Sokolov and Ternov, the probability of spin-flip. In this case, the asymptotic polarisation,

$P_{\rm TBK}$, is, in fact, given by

$$P_{\rm TBK} = 100\% \tag{6.137}$$

(antiparallel to the field in the case of a neutron), and the characteristic polarisation time is given, for motion transverse to the field, by

$$\tau_{\rm TBK} = \left\{ \frac{64}{3} \frac{|\mu|^5 B^3 \gamma^2}{4\pi\hbar^4} \right\}^{-1}, \tag{6.138}$$

where μ is the magnitude of the magnetic dipole moment, which is positive (negative) if $\boldsymbol{\mu}$ is parallel (antiparallel) to $\boldsymbol{\sigma}$.

6.10.3 Lyuboshitz's simple derivation

Shortly after the work of Ternov, Bagrov and Khapaev was published, Lyuboshitz [141] noted that the results (6.137) and (6.138) did not require a long a cumbersome derivation at all, but rather could be obtained on extremely elementary grounds. His argument was as follows: In a magnetic field, the neutron has an interaction energy, in its rest frame, of

$$-\boldsymbol{\mu}\cdot\boldsymbol{B}.$$

The probability per unit time of a spontaneous M1 transition from the upper to the lower energy state is

$$w = \frac{1}{4\pi} \frac{4}{3\hbar} \omega^3_{\uparrow\downarrow} \left| \langle\downarrow|\boldsymbol{\mu}|\uparrow\rangle \right|^2, \tag{6.139}$$

where the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are parallel and antiparallel to the magnetic field respectively (if $\mu < 0$ is negative, as is usually the case; these states are reversed if $\mu > 0$), and where

$$\hbar\omega_{\uparrow\downarrow} \equiv \Delta E_{\uparrow\downarrow} = |2\mu B| \,. \tag{6.140}$$

Using (6.140) in (6.139), Lyuboshitz thus found

$$P_t(t) = -\left(1 - e^{-t/\tau_{\rm rest}}\right), \tag{6.141}$$

where the characteristic polarisation time, in the rest frame of the neutron, is

$$\tau_{\text{rest}} = \left\{ \frac{64}{3} \frac{|\mu|^5 B^3}{4\pi\hbar^4} \right\}^{-1}, \tag{6.142}$$

To boost this result to the case of arbitrary velocity, one simply needs to recall that time-derivatives (and hence probability rates) are reduced by a factor of γ, and the transverse magnetic field is increased by a factor of γ; substituting these results into (6.142) thus yields an overall factor of $\gamma^3/\gamma = \gamma^2$, and hence the Ternov–Bagrov–Khapaev result (6.138) is reproduced exactly.

Lyuboshitz further noted that, if one wished to apply his argument to the case of *charged* particles with magnetic dipole moments, then one faces the difficulty that the appropriate "rest frame" is, in fact, *accelerated*, and is hence not a Lorentz frame. Clearly, a more general method of attack would be required.

However, if one ignored this complication, to see, as a rough guide, what the Ternov–Bagrov–Khapaev results *would* look like if naïvely applied to an electron, then one may simply replace μ and B by means of the relations

$$\begin{aligned} |\mu| &= \frac{ge\hbar}{4m}, \\ |B| &= \frac{m\gamma}{eR}; \end{aligned} \tag{6.143}$$

one then obtains

$$\tau \sim \left\{ \frac{2}{3} \left| \frac{g}{2} \right|^5 \frac{e^2\hbar\gamma^5}{4\pi m^2 R^3} \right\}^{-1}. \tag{6.144}$$

For the pure Dirac electron, of $g = 2$, one finds

$$\tau_{\text{electron}} \sim \left\{ \frac{2}{3} \frac{e^2\hbar\gamma^5}{4\pi m^2 R^3} \right\}^{-1}. \tag{6.145}$$

The rough result (6.145) gives the *same* dependence on all physical quantities as the Sokolov–Ternov result (6.134); the only difference is that the numerical coëfficient is 2/3, not $5\sqrt{3}/8$. As to the asymptotic polarisation, the rough analysis would probably hazard a guess of simply the neutron result (6.137), namely, 100%; this is close to, but not quite the Sokolov–Ternov result of $\sim 92.4\%$.

Considering the approximations made, this rather heuristic explanation stacks up quite well.

6.10.4 Jackson's comments

In his review article, Jackson [114] describes the Lyuboshitz elementary description of the Ternov–Bagrov–Khapaev effect, and examines in detail why it cannot be pushed too far for charged particles.

Firstly, it is noted by Jackson that, for $g = 2$, the Thomas–Bargmann–Michel–Telegdi precession frequency and the orbital revolution frequency are *equal*; but if g is widely different from 2, then in ultra-relativistic motion the precession frequency is *much higher* than the orbital frequency. Making the quantum-mechanical connection between frequencies and energies, one can then already see that the characteristic energy of the orbital motion is *negligible* compared to that of the magnetic field interaction energy for large g, but is of the *same magnitude* when $g \sim 2$. Since a naïve application of the Lyuboshitz result essentially considers the magnetic energy levels as *isolated*, we should *expect* it to fail when $g \sim 2$.

Secondly, it is noted by Jackson that the Derbenev–Kondratenko [67] results for the Sokolov–Ternov effect for *arbitrary* g show a very sensitive dependence on the value of g—indeed, the direction of polarisation is *reversed* for $0 < g < 1.2$. A naïve application of the Lyuboshitz result (6.144), on the other hand, yields simply a $|g|^5$ dependence. But Jackson notes that, for large $|g|$, the Derbenev–Kondratenko results *do indeed* approach the $|g|^5$ dependence of (6.144).

The remainder of Jackson's paper considers the Sokolov–Ternov effect in detail, for *arbitrary* values of g, by starting with a review of the elementary second-quantisation of the electromagnetic field. The spin-flip synchrotron radiation expressions are obtained, as are the spin-flip rates themselves; the results are in accord with those of Derbenev and Kondratenko. We shall not go into the details here, but merely refer the interested reader to the review article [114].

6.10.5 Classical analysis

Clearly, if one wishes to change the direction of the spin of a particle—classically—then one must do so through the torque equation of motion. An examination of the result (6.130) of Section 6.8.9 reveals that there are not, in fact, any terms in $\boldsymbol{N}_{\text{self}}$ dependent on both q and μ, apart from the spin renormalisation term; but there *are* terms involving μ^2. The finite terms are, in the instantaneous rest frame, given by

$$\boldsymbol{N}_{\text{self}} = \frac{1}{3}\frac{\mu^2}{4\pi}\boldsymbol{\sigma}\times\left\{2\ddot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\times(\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}})\right\}. \tag{6.146}$$

Let us first concentrate on the Ternov–Bagrov–Khapaev effect. In the rest frame of the neutron, its acceleration and jerk are completely negligible (they would be exactly zero, but for the forces of Chapter 4, and those of radiation reaction itself), and hence we may ignore the second term in (6.146). We are therefore left with

$$\boldsymbol{N}_{\text{RR}} = \frac{2}{3}\frac{\mu^2}{4\pi}\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}}. \tag{6.147}$$

Now, from the Thomas–Bargmann–Michel–Telegdi spin precession equation, we have, in the rest frame

$$\boldsymbol{N}_{\text{TBMT}} = \boldsymbol{\mu}\times\boldsymbol{B};$$

hence, we successively find

$$\dot{\boldsymbol{\sigma}} = \frac{\mu B}{s}\boldsymbol{\sigma}\times\hat{\boldsymbol{B}}, \tag{6.148}$$

$$
\begin{aligned}
\ddot{\boldsymbol{\sigma}} &\equiv d_t \dot{\boldsymbol{\sigma}} \\
&= \frac{\mu B}{s} \dot{\boldsymbol{\sigma}} \times \hat{\boldsymbol{B}} \\
&= \left(\frac{\mu B}{s}\right)^2 (\boldsymbol{\sigma} \times \hat{\boldsymbol{B}}) \times \hat{\boldsymbol{B}}, \\
&= -\left(\frac{\mu B}{s}\right)^2 \Big\{ \boldsymbol{\sigma} - (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}}) \hat{\boldsymbol{B}} \Big\}, \qquad (6.149) \\
\dddot{\boldsymbol{\sigma}} &\equiv d_t \ddot{\boldsymbol{\sigma}} \\
&= -\left(\frac{\mu B}{s}\right)^2 \Big\{ \dot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{B}}) \hat{\boldsymbol{B}} \Big\} \\
&= -\left(\frac{\mu B}{s}\right)^3 \boldsymbol{\sigma} \times \hat{\boldsymbol{B}}, \qquad (6.150)
\end{aligned}
$$

where s is the spin of the particle. Using (6.150), the classical radiation reaction torque (6.147) may then be written

$$
\begin{aligned}
\boldsymbol{N}_{\mathrm{RR}} &\equiv \frac{2}{3} \frac{\mu^2}{4\pi} \boldsymbol{\sigma} \times \dddot{\boldsymbol{\sigma}} \\
&= -\frac{2}{3} \frac{\mu^5 B^3}{4\pi s^3} \Big\{ \hat{\boldsymbol{B}} - (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}}) \boldsymbol{\sigma} \Big\}. \qquad (6.151)
\end{aligned}
$$

The directional term in braces shows that the torque is in the *direction of the magnetic field* (the second term simply ensuring that the torque remains perpendicular to $\boldsymbol{\sigma}$). To evaluate its effect, we employ again the transverse polarisation P_t of (6.131):

$$
P_t \equiv (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}});
$$

differentiating this relation, we find

$$
\begin{aligned}
d_t P_t(t) &\equiv d_t (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{B}}) \\
&= (\dot{\boldsymbol{\sigma}} \cdot \hat{\boldsymbol{B}}) \\
&= \frac{1}{s} (\boldsymbol{N}_{\mathrm{RR}} \cdot \hat{\boldsymbol{B}}) \qquad (6.152)
\end{aligned}
$$

Substituting (6.151) into (6.152), we thus find the evolution equation of motion for $P_t(t)$:

$$\dot{P}_t(t) = -\frac{2}{3}\frac{\mu^5 B^3}{4\pi s^4}\left\{1 - P_t^2(t)\right\}. \tag{6.153}$$

This differential equation is nonlinear, but fortunately its general solution is simple:

$$P_t(t) = -\tanh\frac{t - t_0}{\tau_{\text{cl.}}}, \tag{6.154}$$

where

$$\tau_{\text{cl.}} \equiv \left\{\frac{2}{3}\frac{\mu^5 B^3}{4\pi s^4}\right\}^{-1}, \tag{6.155}$$

and where the arbitrary constant t_0 in (6.154) specifies the initial polarisation:

$$P_t(0) \equiv \tanh\frac{t_0}{\tau_{\text{cl.}}}. \tag{6.156}$$

Before we compare these classical results with the spontaneous radiation slip-flip results of the previous sections, we shall first briefly indicate how the classical calculations generalise for the Sokolov–Ternov effect. Because of the acceleration of the electron, an evaluation of the effect in the *rest* frame is not feasible (see the detailed discussion of this problem in Jackson's review [114]); to analyse it in the *lab* frame, we first write the torque equation of motion (6.146) in covariant terms, using the expressions listed in Section G.2.2:

$$(\dot{S}) = \frac{2}{3}\mu^2\eta_0\, U \times \Sigma \times \left\{(\dddot{\Sigma}) + \dot{U}^2(\dot{\Sigma})\right\}. \tag{6.157}$$

After some algebra, one can then show that

$$\begin{aligned} \dot{\boldsymbol{\sigma}} &= \boldsymbol{\sigma} \times \left\{\frac{1}{\gamma}\boldsymbol{C} - \frac{1}{\gamma+1}C^0\boldsymbol{v}\right\} + \dot{\boldsymbol{\sigma}}_T \\ &\equiv \boldsymbol{\sigma} \times \boldsymbol{\Omega}_{\text{RR}} + \dot{\boldsymbol{\sigma}}_T, \end{aligned} \tag{6.158}$$

where $\dot{\boldsymbol{\sigma}}_T$ is the Thomas precession contribution to $\dot{\boldsymbol{\sigma}}$, and

$$C \equiv \frac{2}{3}\frac{\mu^2}{s}\eta_0\left\{(\dddot{\Sigma}) + \dot{U}^2(\dot{\Sigma})\right\}.$$

Now, in this first attack on the problem, we shall simply *ignore* the contribution of Thomas precession terms, due to the radiation reaction *force* equation; from Jackson's discussion [114], we expect that this decoupling of the orbital degrees of freedom will lead to approximate, not exact, results. The calculation of $\boldsymbol{\Omega}_{\rm RR}$ from (6.158), using the electron's Thomas–Bargmann–Michel–Telegdi motion as the zeroth approximation, is rather lengthy, and has been relegated to the computer program KINEMATS, in Section G.4.12; after some manipulation of the results, one finds that the evolution of the transverse polarisation $P_t(t)$ is functionally *identical* to (6.154), with the classical characteristic polarisation time now given by

$$\tau_{\rm cl.} = \left\{ \frac{2}{3} \left| \frac{g}{2} \right|^3 \left(\left| \frac{g}{2} \right|^2 + v^2 \right) \frac{e^2 \hbar \gamma^5 v^3}{4\pi m^2 R^3} \right\}^{-1}. \tag{6.159}$$

If we take the limit of large $|g|$, use expressions (6.143) to convert the circular orbit back into a linear one, and trivially boost back to the rest frame, we are returned to the result (6.155) for the classical stationary neutron, as would be expected [114].

We now compare these completely classical results to those of the previous sections. Firstly, we note that for *spin-half* particles,

$$s = \frac{1}{2}\hbar,$$

the characteristic time, $\tau_{\rm cl.}$, of (6.155) becomes

$$\tau_{\rm cl.}\big|_{s=\frac{1}{2}\hbar} = \left\{ \frac{32}{3} \frac{\mu^5 B^3}{4\pi \hbar^4} \right\}^{-1}.$$

This is exactly *half* of the Ternov–Bagrov–Khapaev result (6.142). It may be thought that the missing factor of 2 could simply be due to an algebraic oversight on the part of the author; it will shortly be clear that this is not the case. Moreover, this factor of 2 is, in fact, the *least* of our concerns: more importantly, the *hyperbolic tangent* in the rate of polarisation, (6.154), is *not* equivalent to the *exponential* rate of (6.141).

The difference between the classical and quantum field theoretical calculations is most dramatic for a neutron initially polarised *in the direction of the field*: the quantum result gives simply $1/\tau_{\text{TBK}}$ as the probability per unit time of it flipping its spin; but the classical result (6.153) *vanishes* as the polarisation approaches $\pm 100\%$.

This behaviour can, in fact, be understood by recalling the assumptions made as to the *nature* of the radiation being considered in each analysis. A stationary neutron with its spin parallel to the magnetic field does not precess; it merely sits there. Classically, if the neutron does not precess, and its does not move, then it *cannot* possibly radiate, since it is simply a static system. If it does not radiate, then clearly there cannot be any effects of radiation reaction. The neutron therefore remains in this static state (albeit perturbatively unstable).

Quantum field theoretically, however, the electromagnetic field is considered to be *quantised into photons*. One therefore has the phenomenon of *spontaneous* radiation coming into play. Classically speaking, with spontaneous radiation the neutron effectively "anticipates" the radiation that it will emit in making the quantum jump from spin-up to spin-down; the reaction from this "anticipated" radiation is what, roughly speaking, causes the neutron to flip its spin—and hence emit the anticipated radiation.

This manifestation of the photon's discreteness would, of course, become proportionally less important if the spin of the particle, s, were large compared to the spin of the photon, $\hbar$, since then each radiated photon would only "step" the spin vector of the particle by the then relatively small amount of $\hbar$. Of course, for a spin of $s = \frac{1}{2}\hbar$, such a step represents a transition from full polarisation to full antipolarisation!—and hence the discreteness of the photon is maximally manifested. Thus, while we have, in the previous chapters of this thesis, successfully dispelled some of the myths of the "large quantum number" rule, as far as spin is concerned, we now finally find its *correct* point of insertion into our conceptual frameworks.

An heuristic argument may be given that explicitly shows this transition from the classical theory, to the first-quantised theory, and then to the second-quantised theory, for the current example, which we shall now outline. Let us return to the classical radiation reaction equation of motion for the spin of the stationary neutron, from equation (6.146):

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \times \boldsymbol{\Omega}_{\mathrm{RR}}, \tag{6.160}$$

where

$$\boldsymbol{\Omega}_{\mathrm{RR}} = \frac{2}{3} \frac{\mu^2}{4\pi s} \ddot{\boldsymbol{\sigma}}. \tag{6.161}$$

To first make the transition to the first-quantised theory, for a spin-half particle, we need to introduce the rest-frame two-spinor

$$|\psi\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} \tag{6.162}$$

into (6.160), in some way. From Ehrenfest's theorem, we know that *expectation values* of operators should be describable classically. Now, for the *left*-hand side of this equation, we know, from our experience with the Thomas–Bargmann–Michel–Telegdi equation, that this generalisation is simply

$$d_t \langle\psi|\boldsymbol{\sigma}|\psi\rangle \, ,$$

namely, the *time-derivative of the expectation value of the spin* is the quantity described by the classical equations. For the *right*-hand side of (6.160), however, we must be more careful. Since this equation describes *radiation reaction*, we in fact require the coöperation of the spin-half particle *twice*: once to *emit* the radiation, and then once again to *receive* the effects of this radiation—as with the classical derivation of this chapter; this is why there are *two* factors of $\boldsymbol{\mu}$ present. Now, if we are to involve the spin-half particle *twice*, then it clearly will not do to form a *single* expectation value using the spinor $|\psi\rangle$; rather, we should expect *two* expectation values to appear. Since the two cross-producted factors in (6.160) each involve $\boldsymbol{\mu}$ once, it may be reasonable to guess that the expectations should be placed around each of these factors separately:

$$d_t\langle\psi|\boldsymbol{\sigma}|\psi\rangle \stackrel{?}{=} \langle\psi|\boldsymbol{\sigma}|\psi\rangle\times\langle\psi|\boldsymbol{\Omega}_{\mathrm{RR}}|\psi\rangle \,. \tag{6.163}$$

Using the expression (6.161) for $\boldsymbol{\Omega}_{\mathrm{RR}}$, and (6.150) for $\dot{\boldsymbol{\sigma}}$, in (6.163), we then have

$$d_t\langle\psi|\boldsymbol{\sigma}|\psi\rangle \stackrel{?}{=} -\frac{2}{3}\,\frac{\mu^2}{4\pi s}\left(\frac{\mu B}{s}\right)^3\langle\psi|\boldsymbol{\sigma}|\psi\rangle\times\Big(\langle\psi|\boldsymbol{\sigma}|\psi\rangle\times\hat{\boldsymbol{B}}\Big)\,. \tag{6.164}$$

The guessed equation of motion (6.164) is, of course, equivalent to the classical result (6.151), with the standard connection

$$\boldsymbol{\sigma}_{\mathrm{classical}} \longleftrightarrow \langle\psi|\boldsymbol{\sigma}|\psi\rangle \,,$$

and will hence lead to the hyperbolic tangent dependence of the polarisation equation (6.154).

So far, we have not changed the net results at all: by going to the first-quantised theory, we still have the same classical radiation reaction behaviour. We now take due note of the *second quantisation* of the photon field. The most important feature that this introduces is that the interaction leading to (6.164) must satisfy *conservation of energy, momentum and angular momentum*. Since the photon has spin $\hbar$, the interaction must, to satisfy conservation of angular momentum, *flip the spin* of the particle. As it stands, however, the expression (6.164) assumes the radiation reaction to be brought about by the *continuous* emission of radiation: it calculates the matrix element from any given state to the *same* state (or one infinitesimally close to it). Thus, to include the effects of the quantisation of the photon, we need to insert *spin-flipped* states into (6.164); the resulting transition probability is then given by the correct formula (6.139). The additional factor of

2, over and above that of the classical quantity (6.155), comes about from the squared three-vector matrix element

$$\left|\langle\downarrow|\boldsymbol{\sigma}|\uparrow\rangle\right|^2 = 2.$$

We thus see that, to obtain the correct physics, we need to use the appropriate *classical* operator, sandwiched between the spinor states of the *first-quantised* theory, which are themselves restricted by the conservation requirements of the *second-quantised* photon theory. It is perhaps fitting that our deliberations should conclude with the three generations of Theoretical Physics congregated together, hand in hand, posing for a final family snapshot.

Appendix A

Notation and Conventions

I sat at Bohr's side during a colloquium at Princeton. The subject was nuclear isomers. As the speaker went on, Bohr got more and more restless and kept whispering to me that it was all wrong. Finally, he could contain himself no longer and wanted to raise an objection. But after having half-raised himself, he sat down again, looked at me with unhappy bewilderment, and asked, 'What is an isomer?'

—— A. Pais, on Niels Bohr [163]

A.1 Introduction

In this appendix, we specify the notations used and conventions followed throughout this thesis.

Some specifications agree with standard physics practice; some simply reflect the eccentricities of the author. It may be taken as granted that the author believes the choices listed below to be most appropriate for investigations in this field. Regardless of their merits, it is hoped that the author has followed each specification consistently and unambiguously.

Readers of Bohr's stature need not take note of this appendix.

A.2 Physical format of this thesis

This thesis is available in both paper and digital formats.

A.2.1 Paper copies

A paper copy of this thesis may be obtained by requesting the author.

The requester may choose the spelling and punctuation conventions to be used in the printing of the thesis; see Section A.3.2 for further details.

Paper copies of this thesis supplied by the author are printed on acid-free paper, and professionally bound.

The first page of this thesis after the title page is numbered page 1; each subsequent page is numbered consecutively, without interpolations or omissions. The final page of this thesis is page number 574.

A.2.2 Digital copies

This thesis is also available, complete, in digital form. It consists of fifty-nine plain ASCII text files, categorised as follows:

The twenty files listed in Table A.1 are LaTeX source files which are self-contained: they may be processed individually (*see* Sections A.2.3, A.2.4 and G.3).

The thirteen files listed in Table A.2 are also LaTeX source files, but they may not be processed individually (*see* Section A.2.3).

The two files listed in Table A.3 are plain ASCII text files, that are the output of the programs CHECKRS and TEST3INT (*see* Section G.3).

The twenty-four files listed in Table A.4 are specially constructed files that are able to serve as both standard ANSI C source files (*see* Chapter G), as well as being able to be included in LaTeX documents via the use of special macros designed by the author. (However, because of their excessive length—over 700 pages, even in a tiny font—the ANSI C source files are not included

Filename	Description
BIBLIOG.TEX	Bibliography.
CLASPCLE.TEX	Chapter 2: Classical Particle Mechanics.
COMPALG.TEX	Appendix G: Computer Algebra.
CRS.TEX	Output of program CHECKRS.
DIPOLE.TEX	Chapter 4: Dipole Equations of Motion.
INTERLAG.TEX	Appendix E: The Interaction Lagrangian.
KM.TEX	Output of program KINEMATS.
NOTATION.TEX	Appendix A: Notation and Conventions.
OVERVIEW.TEX	Chapter 1: Overview of this Thesis.
PUBPAPER.TEX	Appendix F: Published Paper.
RADREACT.TEX	Chapter 6: Radiation Reaction.
RETFIELD.TEX	Chapter 5: The Retarded Fields.
RIGBODY.TEX	Chapter 3: Relativistically Rigid Bodies.
RF.TEX	Output of program RETFIELD.
RR.TEX	Output of program RADREACT.
SUPPID.TEX	Appendix B: Supplementary Identities.
SUPPPRF.TEX	Appendix C: Supplementary Proofs.
THESIS.TEX	Complete thesis.
T3.TEX	Output of program TEST3INT.
VERIFYRF.TEX	Appendix D: Retarded Fields Verification.

Table A.1: The twenty LaTeX source files, included in digital copies of this thesis, that may be processed individually.

Filename	Description
ABSTRACT.TEX	Abstract page.
ACK.TEX	Acknowledgments section.
AMERICAN.TEX	American spelling and punctuation choices.
BRITISH.TEX	British spelling and punctuation choices.
COSTELLA.TEX	The author's spelling and punctuation choices.
CPYRIGHT.TEX	Copyright notice.
DEBUGSPL.TEX	Spelling and punctuation debug mode file.
KMOUTTH.TEX	Output of program KINEMATS.
MACROS.TEX	Thesis macro set used in all LaTeX source files.
QUOTE.TEX	Dirac quotation.
RFOUTTH.TEX	Output of program RETFIELD.
RROUTTH.TEX	Output of program RADREACT.
TITLE.TEX	Title page.

Table A.2: The other thirteen LaTeX source files, included in digital copies of this thesis, that may *not* be LaTeXed on their own.

Filename	Description
CRSOUTTH.TXT	Output of program CHECKRS.
T3OUTTH.TXT	Output of program TEST3INT.

Table A.3: The two plain ASCII text files included in digital copies of this thesis.

Filename	Description
ALGEBRA.H	Header file for computer algebra library.
ALGEBRA1.C	Computer algebra library functions.
ALGEBRA2.C	Computer algebra library functions.
ALGEBRA3.C	Computer algebra library functions.
ALGEBRA4.C	Computer algebra library functions.
ALGEBRA5.C	Computer algebra library functions.
ALGEBRA6.C	Computer algebra library functions.
CHECKRS.C	Source file for program CHECKRS.
FRACTION.C	Fraction library functions.
FRACTION.H	Header file for fraction library.
KINEMATS.C	Source file for program KINEMATS.
KINEMATS.H	Header file for program KINEMATS.
LATEXOUT.C	LaTeX output library functions.
LATEXOUT.H	Header file for LaTeX output library.
MISCUTIL.H	Header file with miscellaneous utilities.
RADREAC1.C	Source file for program RADREACT.
RADREAC2.C	Source file for program RADREACT.
RADREAC3.C	Source file for program RADREACT.
RADREAC4.C	Source file for program RADREACT.
RADREAC5.C	Source file for program RADREACT.
RADREACT.H	Header file for program RADREACT.
RETFIELD.C	Source file for program RETFIELD.
RETFIELD.H	Header file for program RETFIELD.
TEST3INT.C	Source file for program TEST3INT.

Table A.4: The twenty-four ANSI C source code files included in digital copies of this thesis.

in paper copies of this thesis).

It should be noted that, to process the *complete* text of this thesis, one requires all of the ASCII files listed in Tables A.1, A.2 and A.3 (*see* Section A.2.3), but *not* those listed in Table A.4, which are only required if one wishes to run the computer algebra programs oneself (*see* Appendix G). Individual chapters and appendices may also, if desired, be processed individually (*see* Section A.2.4).

All of the fifty-nine ASCII files consist of printable ASCII characters only (ASCII codes 32 through 126), plus end-of-line characters (the precise number and values of which are machine-dependent). LaTeX source files have lines with lengths not exceeding 70 characters (excluding end-of-line characters); ANSI C source files have lines with lengths not exceeding 77 characters (excluding end-of-line characters); *no* file has the word `From` appearing at the start of any line (which is corrupted in the Internet e-mail system to `>From`).

A.2.3 Processing the complete thesis

To process the complete text of this thesis, one requires a digital copy of the thesis, and a computer system installed with a "big" implementation of the TeX and LaTeX document preparation systems. Note that "standard" implementations of TeX and LaTeX (typically having of the order of 65,000 words of main memory) are *not* sufficiently powerful to process this complete thesis—nor, indeed, a number of its constituent chapters.

The ASCII source files listed in Tables A.1, A.2 and A.3 must be in a single directory on the computer system.

One must then simply LaTeX the file THESIS.TEX to create the DVI file for the complete thesis. (It is necessary to do this several times, to ensure the cross-references are correctly linked, and that the Table of Contents is included in its final form.)

LaTeX will ask the user to input the spelling convention desired: either

COSTELLA, BRITISH or AMERICAN must be entered by the user at the prompt (*see* Section A.3.2).

Upon completion of processing by LaTeX, one may then either view the output electronically with a DVI viewer, or print any or all pages of the thesis using the DVI software available on one's local system. Note that the resolution of the fonts used for viewing or printing *must* be 300 dots per inch for the output to be as designed by the author; resolutions other than this may produce spurious artifacts, and do *not* represent authorised copies of this thesis.

It should be noted that some digital copies of this thesis include a complimentary copy of the binary file THESIS.DVI, which includes the processed output of the complete thesis, using the author's spelling and punctuation conventions. If one is in possession of such a copy, one need not carry out the LaTeXing instructions listed above; the file THESIS.DVI may be viewed or printed immediately.

A.2.4 Processing a single chapter or appendix

To process only a *single* chapter or appendix of this thesis, the TeX and LaTeX document processing systems must be installed on a computer system. Some chapters and appendices only require a "standard" implementation of these systems, but others require a "big" implementation; the dividing line is implementation-dependent, so one can only find out by trying the desired chapter or appendix on one's own system.

The files listed in Table A.1 may be processed individually. Note that the other .TEX files, listed in Table A.2, may *not* be processed individually: they only appear in the complete thesis.

To process the chapter or appendix in question, one usually only requires the .TEX file for that particular chapter or appendix, *plus* the thesis macro package file MACROS.TEX, and one of the three spelling files COSTELLA.TEX, AMERICAN.TEX or BRITISH.TEX. (The exception is the appendix COMPALG.-TEX, which also requires the presence of the files KMOUTTH.TEX, RFOUTTH.-TEX, RROUTTH.TEX, T3OUTTH.TXT and CRSOUTTH.TXT.) These files must reside in a single directory of the computer system.

One then simply needs to LaTeX the file corresponding to the chapter or appendix in question (several times, to link the cross-references correctly).

The files have been constructed so that *no* errors or warnings are generated on the final pass of LaTeX. Cross-references to sections, subsections or equations in other chapters or appendices, or to entries in the Bibliography, are replaced with harmless 'x' characters, rather than generating an error. Cross-references to sections, subsections and equations *within* the chapter or appendix in question are processed normally.

Page numbering is initialised at the beginning of the chapter or appendix; however, the Chapter and Section numbers take the values that they do in the complete thesis.

For convenience, a Table of Contents, covering only the chapter or appendix in question, is added to the output. (In the complete thesis, only a single complete Table of Contents is included, before the first chapter.)

A.3 Language and typography

A.3.1 Document preparation system

This thesis has been prepared using the LaTeX document preparation system [129].

The default text type is 12-point Computer Modern, 1.5 spaced. All fonts, styles, commands and macro constructions used in this thesis follow the guidelines given in the LaTeX manual [129].

This thesis has been designed to be viewed or printed using standard LaTeX fonts of a resolution of precisely 300 dots per inch; reproductions using fonts of other resolutions, or fonts that are not those standard to LaTeX, do not represent authorised copies of this thesis.

A.3.2 Dialect-independent English

This thesis is written in dialect-independent English: it may be printed with either British English, American English or the author's English conventions; the printing you are reading uses the author's conventions. (All official copies submitted for examination use the author's conventions.)

Diæreses are used to separate nondiphthonged adjacent vowels. Nouns collected together into the one entity are separated by en dashes. Some words of foreign origin have their original accenting retained. Hyphens whose omission would not lead to ambiguity, ugliness or a violation of an accepted custom are omitted. Foreign phrases not naturalised into the English language are generally italicised. Latin abbreviations are italicised and punctuated. Terms are generally italicised when they are introduced or defined. Italicisation is also used for emphasis in the text. Foreign names are reproduced as faithfully as possible, following the printed spelling in the source literature.

Some of the above specifications do not apply to the published paper included verbatim in Appendix F.

Words and phrases used by the author of arguably questionable spelling, grammar or semantics have been checked by the author with the full Oxford English Dictionary [161]. Typographical and other errors may of course still be present in the text of this thesis.

A.3.3 Political incorrection

The author does not belong to the Political Incorrection movement.

Readers of that persuasion are warned that phrases of an offensive nature may be found within this thesis. For example, quantities are referred to as *dimensionless*, rather than *dimensionally challenged*; erroneous results are referred to as *incorrect*, rather than *fictionally gifted*; and so on.

Readers for whom such terminology would be psychologically challenging may request a digital copy of this thesis, and digitally censor the offending phrases themselves.

A.3.4 Units

The base system of units for this thesis is the SI system.

Theoretical investigations are carried out in the system of units derived from the SI system when they are "naturalised" according to

$$c = \varepsilon_0 = \mu_0 = 1. \tag{A.1}$$

A.3.5 Decimal separator

The decimal separator is the full stop, as in the value 137.036.

A.3.6 Standard symbol set

The *standard symbol set* is the set of typographical symbols that are used in this thesis to represent general mathematical quantities. Any symbol from the standard symbol set is referred to as a *standard symbol*.

Some classes of mathematical quantities are, as described in the following sections, denoted in a unique, identifiable way. This may be done by typographically modifying a standard symbol's typeface, or attaching an extra mark to it (collectively referred to as *adorning* the symbol); or it may be performed by using symbols not in the standard symbol set. All of the above specialised forms of notation are referred to as *special symbols*. Special symbols are not a part of the standard symbol set.

All mathematical quantities not denoted by special symbols are denoted by symbols from the standard symbol set. There is no way to distinguish, typographically, the mathematical nature of such quantities. The nature of the quantity will either be explicitly spelt out in the text, or it will be implied by context.

The standard symbol set consists of the following symbols: a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q, r, s, t, u, v, w, x, y, z, A, B, C, D, E, F, G, H, I, J, K, L, M, N, P, Q, R, S, T, U, V, W, X, Y, Z, α, β, γ, δ, ε, ζ, η, θ, ϑ, κ, λ, μ, ν, ξ, π, ρ, σ, τ, ϕ, φ, χ, ψ, ω, $\varGamma$, $\varDelta$, $\varTheta$, $\varLambda$, $\varXi$, $\varPi$, $\varSigma$, $\varUpsilon$, $\varPhi$, $\varPsi$ and $\varOmega$.

It will be noted that all standard symbols are italicised. All typographical modifications to the standard symbol set used in this thesis retain the italicisation of the standard symbol part of the modified symbol.

A symbol is referred to as a *collection symbol* if it represents a collection of quantities rather than a single quantity; these quantities are referred to as being *housed* in the collection symbol. The act of typographically modifying a collection symbol to denote the extraction of a quantity housed within it is referred to as *dereferencing* the collection symbol.

A.3.7 Decommissioning and recommissioning

If a symbol, word or phrase is explicitly specified as *decommissioned* by the author, then it is thenceforth deemed to be an invalid symbol, word or phrase for the purposes of the remainder of this thesis, even if it would otherwise be a valid construct according to the rules laid down in this appendix.

Decommissioned symbols, words or phrases may only be used for the purpose of *recommissioning*. If a symbol, word or phrase is recommissioned by the author, after having been previously decommissioned, then it is thenceforth deemed to again be a valid construct.

If a phrase is recommissioned as *nonmodifiable*, then no typographical symbols may appear in the interior of the phrase, apart from line-breaking hyphenation marks. In particular, additional words may *not* be added between the individual words of a nonmodifiable phrase.

For the purposes of decommissioning and recommissioning, the pages of this appendix are taken to precede all other pages in this thesis. All possible symbols, words and phrases constructable are deemed to be commissioned at the start of this appendix.

The provisions of this section do not apply to the published paper included verbatim in Appendix F.

A.3.8 Precedence of operations

The following is a list of the typographical symbols used in this document to denote operations, in order of decreasing precedence. Operators listed consecutively are of the same precedence; steps down the precedence hierarchy are noted explicitly. Operators of the same precedence are evaluated left-to-right, except where noted. The other sections in this appendix should be consulted for details about the operations listed, the language used to describe them, and their notation.

Binding symbols: ( ), [ ] and { }. These three sets of symbols, referred to as *binding parentheses*, *binding brackets* and *binding braces* respectively, "bind" symbols together to force precedence. Except where otherwise noted, they are in all mathematical respects equivalent, and hence usable interchangeably; however, certain conventions have emerged, æsthetically pleasing and visually simplifying, such as the use of binding parentheses around three-vector dot products. Binding parentheses, brackets and braces are usually used to force the intended precedence when the non-explicitly-bound expression does not suffice, but they may also be used redundantly for the purposes of emphasis or visual clarity.

Symmetrisors: $\{\!\{\ \}\!\}$. These have the same precedence as binding symbols. *See* Section A.6.3.

Primed quantities: $'$, $''$. Quantities that are of like form to an original quantity which has been designated a standard symbol, but which must be distinguished from it, may be given the original standard symbol with one, two or more primes. The prime or primes form part of the new symbol, and bind more tightly than any operator listed below.

The square root symbol: $\sqrt{}$. The quantities appearing underneath the overbar of this symbol are treated as if they lie within an overall binding symbol, and then the *positive* square root of the result is taken.

Matrix indexing: superscripts and subscripts. Matrix indices are deemed to have higher precedence than power-raising superscripts; thus, if x^i is a component of a four-vector, then $x^{i\,2}$ is equivalent to $(x^i)^2$, without need for the explicit binding.

Raising of powers: superscripts. If a superscript does not fall into one of the above categories, and if it is a valid value, then it denotes the raising of the base to the power of the superscript. Powers are evaluated right-to-left (*i.e.*, typographically highest subscript first). If the superscript is *rational* (but not integral), and the base is real and non-negative, the result is defined to be the *positive* root; if the base is not real and non-negative, the result is defined by an explanatory note. If the superscript is irrational, and the base is not a non-negative real, the expression is in error. If $\boldsymbol{A}$ is a three-vector, then $\boldsymbol{A}^{2m} \equiv (\boldsymbol{A} \cdot \boldsymbol{A})^m$. If B is a four-vector, then $B^{2m} \equiv (B \cdot B)^m$.

Time-derivatives: overdots. Overdots are used to denote the proper-time or lab-time derivative of a quantity; see Section A.8.20 for details of how the choice of time is decided. Overdots have *lower* precedence than the symbols above. Thus, $\dot{\boldsymbol{\sigma}}'$ is defined to be $d_t\boldsymbol{\sigma}'$; *i.e.*, the prime on $\boldsymbol{\sigma}'$ binds more tightly than the overdot.

The scientific-notation symbol: $\times\,10$. The notation $x \times 10^n$, where x is real and n is integral, denotes a real number in scientific notation.

Generic multiplication: adjacent symbols. If two symbols are placed next to each other, then they are multiplied together according to the rules appropriate to their particular types.

The wedge-product symbol: $\wedge$. *See* Section A.8.11.

The cross-product symbol: $\times$. The cross-product symbol $\times$ has higher precedence than the dot-product symbol $\cdot$, so that triple-products of the form $\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C}$ are syntactically correct without need to force binding. *See* Sections A.8.10 and A.9.13.

The numeral multiplication symbol: $\cdot$. This symbol is sometimes used to denote multiplication of numerical quantities where the use of adjacent symbols would be ambiguous, such as in $2 \cdot 3$.

The operator-delimiter multiplication symbol: $\cdot$. Usually, multiplication of scalar quantities is signified by simply placing the symbols for the quantities next to each other. However, it is often necessary to delimit the scope of operators, such as derivative operators. While this is least ambiguously performed by the use of binding symbols, one may also delimit the scope of simple operators by the use of the central dot, *if* this usage is unambiguous. Thus, $d_x A \cdot d_y B$ denotes $(d_x A)(d_y B)$. However, this notation cannot be used where it would interfere with other interpretations of the central dot.

The dot-product symbol: $\cdot$. *See* Sections A.8.7 and A.9.11.

The product-continuation symbol: $\times$. This symbol is used to denote the continuation of a typographically long product of factors onto subsequent lines of a multiline equation, in all cases *except* when the multiplication operation in question is the dot-product (see below). In this rôle as a continuation symbol, the symbol $\times$ has no other connotation than that of generic multiplication, of whatever flavour, and has the same precedence as the multiplication operation it represents.

The dot-product continuation symbol: $\cdot$. This symbol is used to continue a typographically long product of factors onto subsequent lines of a multiline equation, if the multiplication operation in question is the dot-product.

The inline fraction bar: /. This symbol is used as the inline equivalent of the fraction bar used in displayed expressions. It may also be used in displayed expressions where typography is improved by its use. It should be noted that the inline fraction symbol / has lower precedence than all multiplication operations, so that abc/xyz is identical to $(abc)/(xyz)$. However, if more than one / appears in any product, without binding indicated, then the expression is in error: the intended binding must be forced.

The addition and subtraction symbols: $+$ and $-$. These symbols have lower precedence than those listed above.

The summation and product symbols: $\sum$ and $\prod$. These symbols have lower precedence than those listed above.

The Einstein summation convention: The presence of two indices with the same standard symbol in a product of matrix factors implies a summation of that factor with the index in question taking on all values of its enumeration set. For three-vectors, the arbitrary position of either index is irrelevant; for four-vectors, however, the convention only applies if one copy of the index is covariant and the other is contravariant. If more than one copy of a standard symbol appears as an index in an expression, and these copies *do not* satisfy these rules, then no summation convention applies; no explicit note need be made. The summation convention may also be suppressed by use of the parenthesised words "(no sum)" at the end of any expression.

The evaluation symbol: $|_{a=b}$. This symbol has lower precedence than all those listed above. *See* equation (A.10) of Section A.3.10 .

The comparison operators: $=$, $\neq$, $>$, $<$, $\geq$, $\leq$, $\not>$, $\not<$, $\not\geq$, $\not\leq$. These symbols have lower precedence than all those listed above.

A symbol used to denote a function is *overloaded* if it is defined a number of times, with a typographically different parameter list for each definition. The nature of the parameter list for any usage of that function symbol then determines which particular function definition applies to that usage of the symbol.

A.3.9 Electric charge

A general electric charge is always denoted by the symbol

$$q \tag{A.2}$$

throughout this thesis.

The symbol q is also used in its common usage as a general Hamiltonian or Lagrangian coördinate degree of freedom; there is no ambiguity in practice.

The charge on the positron is denoted by the symbol e. The charge on the electron is consequently $-e$. The symbol e is also used for other purposes; no confusion arises in practice.

A.3.10 Derivatives

Throughout this thesis, the n-th order total derivative operator with respect to a quantity q,

$$\frac{d^n}{dq^n}, \tag{A.3}$$

is replaced by the notation

$$d_q^n. \tag{A.4}$$

The notation (A.3) is decommissioned. Likewise, the n-th order partial derivative operator,

$$\frac{\partial^n}{\partial q^n}, \tag{A.5}$$

is replaced by the notation

$$\partial_q^n. \tag{A.6}$$

The notation (A.5) is decommissioned.

If the quantity being differentiated with respect to is the spacetime position x^μ or x_μ, then it is deemed that the subscript $_\mu$ or superscript $^\mu$ may be used instead of x^μ or x_μ for the partial derivative:

$$\begin{aligned} \partial_{x^\mu} &\longleftrightarrow \partial_\mu, \\ \partial_{x_\mu} &\longleftrightarrow \partial^\mu. \end{aligned} \tag{A.7}$$

The *d'Alembertian operator* is denoted ∂^2, and is defined as

$$\partial^2 \equiv \partial^\alpha \partial_\alpha \equiv \partial_t^2 - \nabla^2. \tag{A.8}$$

When there are several four-position variables (such as x^μ, x'^μ, x''^μ, *etc.*) in an equation, the d'Alembertian operator is adorned with a subscript indicating the intended variable, such as

$$\partial_x^2,\ \partial_{x'}^2,\ \partial_{x''}^2. \tag{A.9}$$

The standard symbol d is decommissioned. The symbol d is recommissioned as a standard symbol, if and only if the quantity it is to represent does not need to be, or may not be interpreted as, a quantity for which any subscripting operation is valid. Textual labels or identifiers, normally denoted by subscripts, may be placed in parenthesised subscripts for the recommissioned symbol d.

For any function $f(q)$ of q, the notation

$$d_q^n f(q)\Big|_{q'} \tag{A.10}$$

indicates that the n-th order total derivative of $f(q)$ with respect to q is to be taken, and then the result evaluated at the point $q = q'$. The notation is likewise extended to partial derivatives and functions of more than one variable, and is also deemed to apply if overdots are used to denote the taking of the derivative (*see* Section A.8.20).

A.3.11 Energies, momenta, angular momenta

There are two different concepts generally referred to by the term "momentum". The first is that of a *mass-weighted velocity quantity*; it is used almost universally in Newtonian mechanics and General Relativity. In this thesis, this type of quantity is *always* referred to as *mechanical momentum*, and it is *always* denoted by the standard symbol

$$p, \tag{A.11}$$

adorned as appropriate.

The second concept of "momentum" is that of a *conjugate Hamiltonian degree of freedom*; it is used almost universally in Hamiltonian mechanics. In this thesis, this type of quantity is *always* referred to as *canonical momentum*, and is *always* denoted by the standard symbol

$$b, \tag{A.12}$$

adorned as appropriate.

The concept of "energy", being the relativistic timelike component of the four-momentum, must likewise be categorised as either *mechanical energy* or *canonical energy*. Mechanical energy, if not denoted p^0 or p_0, *must* be denoted by the standard symbol

$$W. \tag{A.13}$$

Canonical energy, if not denoted by b^0 or b_0, *must* be denoted by one of the standard symbols

$$E, H. \tag{A.14}$$

The symbol H may only be used if the canonical energy is in fact the *Hamiltonian*.

The concept of "angular momentum", as the product of the momentum of an object with its position relative to some coördinate origin, must also be categorised as either *mechanical angular momentum* or *canonical angular momentum*. Mechanical angular momentum *must* be denoted by one of the standard symbols

$$s,\ S,\ l,\ L,\ j,\ J. \tag{A.15}$$

Canonical angular momentum *must* be denoted by one of the adorned standard symbols

$$\hat{s},\ \hat{S},\ \hat{l},\ \hat{L},\ \hat{\jmath},\ \hat{J}. \tag{A.16}$$

It should be noted that *quantum mechanics* is phrased in terms of Hamiltonian mechanics; thus,

$$b^\mu \equiv i\partial^\mu \tag{A.17}$$

is the appropriate notation for the quantum mechanical canonical momentum operator, in terms of the guidelines above.

Due to the almost universal confusion on this point, the restrictions above on the acceptability of symbols for momenta, energies and angular momenta are deemed to override all other rules in this appendix. In particular, they override the otherwise general rule of Section A.3.6 relating to the typographical equivalence of standard symbols.

For the same reason, the words and phrases *momentum*, *energy* and *angular momentum*, and their plurals, are hereby decommissioned.

The nonmodifiable phrases *mechanical momentum*, *mechanical self-momentum*, *mechanical three-momentum*, *mechanical four-momentum*, *mechanical energy*, *kinetic energy*, *mechanical self-energy*, *mechanical centre of energy*, *mechanical angular momentum*, *mechanical angular self-momentum*, *canonical momentum*, *canonical three-momentum*, *canonical four-momentum*, *canonical energy* and *canonical angular momentum*, and their plurals, are explicitly recommissioned.

The nonmodifiable adjectives *positive-energy* and *negative-energy* are also explicitly recommissioned, but only for purposes for which the canonical energy and mechanical energy are equal in value.

A.3.12 Classical, quantum, relativistic

The adjective *quantum* is used to denote the presence of, or required presence of, single-particle or multiparticle quantum-mechanical considerations. Its antonym is defined to be the adjective *classical*.

The adjective *relativistic* denotes the presence of, or required presence of, the mechanics of Einstein's special theory of relativity. The adjective *nonrelativistic* denotes the use of Galilean mechanics. Some subtle problems involved in making the transition between the fundamentally incompatible structures of relativistic and nonrelativistic mechanics are discussed in Sections 2.6.12 and A.8.25.

It should be noted that the adjectives "classical" and "relativistic" are linearly independent: relativistic classical physics, nonrelativistic classical physics, relativistic quantum physics, and nonrelativistic quantum physics, are separate quadrants of physics.

A.3.13 Quantities

A *physical quantity* is defined to be any aspect of a physical phenomenon that may be described in mathematical terms. A *mathematical quantity* is defined to be a mathematical construct that obeys some well-defined mathematical laws. Mathematical quantities may be used to *represent* physical quantities.

In this thesis, the term *quantity* is used for either of these concepts, not always unambiguously.

A.3.14 Explicit quantities

When the author refers to a quantity as being *explicit*, then it generally means that some elegant yet highly abstract expression has been "expanded out" in terms of components that are generally less æsthetically elegant, but usually more in contact with one's intuitive understanding of the physical world.

This often, but not always, means that a *covariant* expression has been written in terms of *non-covariant* components (*see* Section A.3.18 for a precise definition of these words, according to the author's terminology).

A.3.15 Pointlike objects

If an object has zero or infinitesimal spatial extent, then it is referred to as being *pointlike*.

The term *point* is used as an adjective, identical in meaning to *pointlike*, but it may only be used directly in front of the noun of the object it is describing. For example, a particle of zero extent may equivalently be referred to as a *point particle*, or as a *pointlike particle*; but if the order of the words is reversed, one may only say that the *particle is pointlike*, since in this context the use of the adjective "point" would sound absurd.

The choice of using either *point* or *pointlike*, in cases where both would be permitted, is arbitrarily made.

A.3.16 Densities

If an object carries some physical characteristic Q distributed over a finite volume in $\boldsymbol{r}$-space, the "Q density" of that characteristic may either be denoted

$$\rho_Q(\boldsymbol{r}) \tag{A.18}$$

or

$$Q_\rho(\boldsymbol{r}) \tag{A.19}$$

without need for explicit comment.

The former choice is, historically, more common; but the latter choice is both more visually explicit (with the characteristic Q dominating the symbol, rather than being relegated to a subscript), as well as being often *necessary* when using the standard LaTeX typesetting system—since adornments such as boldfacing may not be used in subscripts.

A.3.17 Orders

For the purposes of this thesis, the terminology surrounding the term *order* is defined as follows:

A term T is said to be "of order $f(A)$", where $f(A)$ is some function f of some quantity A, if, were T to have its (implicit or explicit) dependencies on A written out explicitly, the resulting explicitly-written term would, when simplified, contain a factor of $f(A)$, and no other dependency on A.

If the function $f(A)$ is a power A^n of A, then T may be simply said to be "of order n in A". The power n may, in general, be of arbitrary type, but in practice it is usually integral.

If unambiguous, the term T may be simply said to be "of order n", without explicitly mentioning A, if the preceding context makes it clear that it is the order of A that is being discussed.

An expression E is said to be "of order n" in some quantity A if the lowest (highest) order in A of any of its terms is n; the choice of whether it is the lowest or highest order depends on the application.

An expression E is said to be "expanded to order n in A" if all of the terms in E with order in A less than (greater than) or equal to n are written out explicitly, and the remaining terms—if any—replaced with the symbol

$$+\,\mathrm{O}(A^m), \tag{A.20}$$

where m is the lowest (highest) order in A of the omitted terms. If E contains no terms of higher (lower) order in A than n, then the symbol $+\,\mathrm{O}(A^m)$ shall not be used.

A.3.18 Covariance

Consider the "G" group, where "G" stands for the name of the person or other object that the group is named after. Any arbitrary mathematical quantity that transforms as a representation of the G group is referred to as a *mathematically G-covariant quantity*. Any mathematical quantity that does not transform as a representation of the G group, and which, even if considered together with an arbitrary number of other quantities that do not transform as a representation of the G group, can still never be made to transform as a representation of the G group, is referred to as a *mathematically non-G-covariant quantity*.

Any symbolic representation of a mathematically covariant quantity is referred to as a *manifestly G-covariant quantity*, or, simply, a *G-covariant quantity*, where the adjective "manifest" is used for emphasis or disambiguation. Any symbolic representation of a *subpart* of a G-covariant quantity, that is not *itself* mathematically G-covariant, is referred to as a *non-G-covariant quantity*.

Note that, according to these definitions, for any given non-G-covariant quantity there always exist other non-G-covariant quantities such that, when considered together, the resultant structure is, as a whole, a G-covariant quantity. Thus, non-G-covariant quantities are *never* mathematically non-G-covariant; conversely, mathematically non-G-covariant quantities are never non-G-covariant quantities. All quantities are either G-covariant, non-G-covariant, or mathematically non-G-covariant quantities.

These definitions appear to be counterintuitive, but their application will reveal their usefulness. In particular, the "mathematically" forms of the above definitions are rarely used in this thesis.

When unambiguous, the "G-" prefix in the above may be omitted from a discussion where the group under consideration is understood. In such circumstances, the "G-" may be reïnserted at any point for emphasis or disambiguation.

The adjective "covariant" is also used in its traditional form as a conjugate to "contravariant", when applied to indices. This meaning of the word "covariant" has nothing to do with the definitions above.

A.4 Enumeration sets

An *enumeration set* is defined as a set of consecutive integral values

$$\Big\{\omega,\, \omega+1,\, \omega+2,\, \ldots,\, \omega+d-2,\, \omega+d-1\Big\}. \tag{A.21}$$

The term *enumeration* is used to describe an element of an enumeration set.

The first integral value in an enumeration set—ω in (A.21)—is referred to as the *offset* of the enumeration set. An enumeration set with an offset of zero is referred to as having or being *zero-offset*. An enumeration set with an offset of unity is referred to as having or being *unit-offset*.

The number of integral values in the enumeration set is referred to as the *dimension* of the enumeration set.

Any integral value that is listed in the enumeration set is said to *lie within the bounds* of the enumeration set.

Any operation defined for integers may be used for enumerations, if the operation results in a legal expression. In particular, if the result of an operation on one or more enumerations is itself to be used as an enumeration, then the result must be an enumeration lying within the bounds of the enumeration set.

The *ordered enumeration set* of an enumeration set is the ordered set of the enumerations listed in increasing numerical order.

Any of the enumerations in an enumeration set may, for convenience, be given an alternative non-numerical notation. Such optional pieces of notation are referred to as *enumeration names*. An enumeration name may be the same as other symbols used for other purposes, provided that those other symbols do not themselves take values from the original enumeration set. A given enumeration in an enumeration set may, in general, be given more than one enumeration name. However, within any given enumeration set, any given enumeration name may refer to only one enumeration.

The adjective *enumerated* will be used rather than the adjective *integral* where appropriate.

A.5 Special functions

A.5.1 Kronecker delta function

The *Kronecker delta function* δ^i_j, taking enumerated arguments i and j, is defined as

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{A.22}$$

A.5.2 Dirac delta function

The *Dirac delta function* $\delta(t)$, of real argument t, is defined as

$$\delta(t) \equiv \lim_{\varepsilon \to 0} g(t, \varepsilon), \tag{A.23}$$

where $g(t, \varepsilon)$ is any function in the ensemble of indefinitely differentiable functions of real arguments t and $\varepsilon > 0$ such that $g(t, \varepsilon) \to 0$ for all $t \neq 0$ as $\varepsilon \to 0$, and

$$\int_{-\infty}^{\infty} dt'\, g(t', \varepsilon) = 1. \tag{A.24}$$

This ensemble is referred to as the *Dirac delta ensemble*.

There is only one Dirac delta function in this thesis. This function may, however, be instantiated for an arbitrary number of purposes. The limiting procedure above accompanying its function definition is deemed to be executed on the first blank page following the end of this thesis. If, at that point, the evaluation of any given mathematical expression appearing in this thesis is not invariant under a change of $g(t, \varepsilon)$ through the Dirac delta ensemble, then the expression in question cannot describe a physical quantity.

The *three-dimensional Dirac delta function*, of three-vector argument $\boldsymbol{r}$, is denoted $\delta^{(3)}(\boldsymbol{r})$, or simply $\delta(\boldsymbol{r})$ where unambiguous, and is defined as

$$\delta^{(3)}(\boldsymbol{r}) \equiv \delta(x)\delta(y)\delta(z), \tag{A.25}$$

where $\boldsymbol{r}$ has components (x, y, z).

The *four-dimensional Dirac delta function*, of four-vector argument X, is denoted $\delta^{(4)}(X)$, or simply $\delta(X)$ where unambiguous, and is defined as

$$\delta^{(4)}(X) \equiv \delta(t)\delta(x)\delta(y)\delta(z), \tag{A.26}$$

where X has components (t, x, y, z).

A.5.3 Heaviside step function

The *Heaviside step function* $\vartheta(t)$ is defined as the integral of the Dirac delta function (A.23):

$$\vartheta(t) \equiv \int_{-\infty}^{t} dt'\, \delta(t'). \tag{A.27}$$

The same considerations for evaluation and physicality apply to the Heaviside step function as apply to the Dirac delta function (*see* Section A.5.2).

A.5.4 Alternating function

The *d-dimensional alternating function of offset* ω is denoted

$$\varepsilon_{\{i\}}^{(\omega,d)}. \tag{A.28}$$

Its argument is a special collection symbol $\{i\}$ of enumerations (*see* Section A.4), which may be dereferenced by removing the braces and subscripting with a unit-offset enumeration of dimension d. The collection symbol $\{i\}$ may be also be denoted either by writing its elements adjacently, in order of the enumeration index:

$$\{i\} \equiv i_1 i_2 i_3 \cdots i_{d-2} i_{d-1} i_d, \tag{A.29}$$

or by writing them in set notation:

$$\{i\} \equiv \{i_1, i_2, i_3, \ldots, i_{d-2}, i_{d-1}, i_d\}. \tag{A.30}$$

Each of the i_m take values from the enumeration set of dimension d and offset ω. The function $\varepsilon^{(\omega,d)}_{\{i\}}$ is equal to zero if any $i_m = i_n$ for m and n in this enumeration set and $m \neq n$. Otherwise, the function is equal to $+1$ (-1) if the set (A.30) is an even (odd) permutation of the ordered enumeration set of dimension d and offset ω.

The argument d may be omitted from the notation $\varepsilon^{(\omega,d)}_{\{i\}}$ if the collection symbol $\{i\}$ is explicitly expanded into the notation (A.29) or (A.30), or equivalent enumeration names. These elements i_m are often themselves given enumeration names. The superscript ω is often omitted as being implicitly understood by context.

Of particular use are the three-dimensional unit-offset alternating function, $\varepsilon^{(1)}_{i_1 i_2 i_3}$, and the four-dimensional zero-offset alternating function, $\varepsilon^{(0)}_{i_1 i_2 i_3 i_4}$. The former is generally referred to as simply the *three-dimensional alternating function*, and denoted $\varepsilon_{i_1 i_2 i_3}$, where each i_m takes on values from the unit-offset enumeration of dimension three, *viz.*, $\{1, 2, 3\}$; the latter as the *four-dimensional alternating function*, and denoted $\varepsilon_{i_1 i_2 i_3 i_4}$, where each i_m takes on values from the zero-offset enumeration of dimension four, *viz.*, $\{0, 1, 2, 3\}$. From the above definition of $\varepsilon^{(\omega,d)}_{\{i\}}$, it is clear that

$$\varepsilon_{123} \equiv \varepsilon_{0123} \equiv +1. \tag{A.31}$$

The four-dimensional alternating function's indices may, for the purposes of Lorentz-covariant analyses, be raised by using the fully contravariant metric tensor $g^{\alpha\beta}$, as described in Section A.8.4. The definition (A.31) still remains in effect; thus, $\varepsilon^{0123} \equiv -1$.

A.6 Associativity and commutativity

In this thesis, all addition and multiplication operations for arbitrary objects are assumed to be associative, without need for this to be explicitly noted.

All addition operations for arbitrary objects are assumed to be commutative, without need for this to be explicitly noted. No assumptions about commutativity of multiplication operations are made in the general case.

A.6.1 C-numbers and q-numbers

Quantities that commute with all other quantities are referred to as *c-numbers*.

Any quantity that is not a c-number is a *q-number*.

A.6.2 Commutators and anticommutators

The *commutator* of two quantities A and B is defined to be $AB - BA$, and is denoted by brackets, thus:

$$[A, B] \equiv AB - BA. \tag{A.32}$$

The *anticommutator* of two quantities A and B is defined to be $AB + BA$, and is denoted by braces, thus:

$$\{A, B\} \equiv AB + BA. \tag{A.33}$$

Commutators and anticommutators may be collectively referred to as *permutators*. Quantities that are commutated or anticommutated are collectively referred to as being *permutated*.

Indices may also be permutated:

$$A_{[\alpha} B_{\beta]} \equiv A_\alpha B_\beta - A_\beta B_\alpha \tag{A.34}$$

and

$$A_{\{\alpha} B_{\beta\}} \equiv A_\alpha B_\beta + A_\beta B_\alpha. \tag{A.35}$$

The permutated indices may be separated by an arbitrary number of intervening factors, and may be repeated for identical indices within the one distributed factor; for example,

$$A_{[\alpha}X_{\mu}Y_{\nu}Z_{\tau}\Big(B_{\beta]}+C_{\beta]}\Big)\,. \tag{A.36}$$

A.6.3 Symmetrisors

The *symmetrisor* of a product of n factors A, B, C, $\ldots$, Z is defined to be $1/n!$ times the sum of the terms of all of the $n!$ permutations of the factors, and is denoted

$$\{\!\{ABC\cdots Z\}\!\}\,. \tag{A.37}$$

For example,

$$\begin{aligned}\{\!\{AB\}\!\} &\equiv \frac{1}{2}\Big(AB+BA\Big)\,,\\ \{\!\{ABC\}\!\} &\equiv \frac{1}{6}\Big(ABC+ACB+BAC+BCA+CAB+CBA\Big)\,.\end{aligned}$$

The use of the double-braces ensures complete disambiguation from the use of braces as a binding symbol, since two identical sets of braces, without intervening symbols, would in the latter case be completely redundant. Furthermore, the double-braces used in (A.37) are typographically spaced closer together than is the case when braces are used as binding symbols.

The inclusion of the factor of $1/n!$ in the above definition of the symmetrisor means that it is *not* necessary to identify or count the number of "truly non-commutative" factors in the product: commuting factors may be (trivially) symmetrised over without affecting the numerical result.

Note also that the symmetrisation is a mathematical one, *not* a typographical one. This is of importance for the three-vector cross-product of Section A.9.13, namely,

$$\boldsymbol{A}\times\boldsymbol{B}\equiv\varepsilon_{ijk}A_jB_k; \tag{A.38}$$

the symmetrisor of (A.38) is defined to be the symmetrisor of the *right-hand side* of this definition:

$$\{\!\{\boldsymbol{A}\times\boldsymbol{B}\}\!\}_i\equiv\frac{1}{2}\varepsilon_{ijk}\Big(A_jB_k+B_kA_j\Big)\,. \tag{A.39}$$

To write the result (A.39) in component-free notation, we must relabel the indices in the second term via $j \leftrightarrow k$, and make use of the identity $\varepsilon_{ikj} \equiv -\varepsilon_{ijk}$; thus,

$$\{\!\!\{ \boldsymbol{A} \times \boldsymbol{B} \}\!\!\} \equiv \frac{1}{2}\Big(\boldsymbol{A} \times \boldsymbol{B} - \boldsymbol{B} \times \boldsymbol{A} \Big) . \tag{A.40}$$

Conceptually, the minus sign in (A.40) may be understood by considering the case when the factors in the symmetrisor are *c-numbers*, in which case the various permuted terms must of course be identical: for c-number three vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, we note the permutation identity $\boldsymbol{A} \times \boldsymbol{B} \equiv -\boldsymbol{B} \times \boldsymbol{A}$, and hence (A.40) is the correct identity.

A.7 Matrices

A matrix is, for the purposes of this thesis, a collection object which is conveniently arranged as a regular rectangular array of quantities.

Any mathematical quantity that is, for the purposes of the investigation in question, considered to be represented in matrix form is referred to as a *matricised* quantity. Matricised quantities are denoted by symbols from the standard symbol set.

A.7.1 Rows and columns

Matrices are considered to be constructed of rows and columns; there is a *matrix element* for the conceptual intersection of each row and column.

The number of rows and number of columns possessed by any given matrix are together referred to as the *dimensions* of the matrix; the dimensions are denoted $N_{\text{rows}} \times N_{\text{columns}}$, where N_{rows} is the number of rows and N_{columns} is the number of columns.

A.7.2 Square matrices

Matrices with an equal number of rows and columns are referred to as *square* matrices.

The dimensions of a square matrix are thus the same value; this is referred to as its *dimension*.

A.7.3 Vectors

A matrix with one only row or only one column is referred to as a *vector*, either a *row vector*, or a *column vector*, respectively. One of the dimensions of a matrix that is a vector is thus equal to unity; the other is referred to as *the dimension* of the vector.

Three-vectors and four-vectors are special types of vector, which are discussed in Sections A.8 and A.9. Except for three-vectors, there are no special symbols to denote vectors.

From the above definitions, it follows that the adjective *d-dimensional* may refer to a vector or a square matrix, but no other matrices.

A.7.4 Indices

The position of a matrix element in a matrix is measured by two enumerated quantities, termed *indices*.

A matrix symbol is dereferenced by placing the row and column indices as subscripts on the matrix symbol, with the row index to the left of the column index; the resulting notation may optionally be placed in parentheses for emphasis.

The enumeration set used to represent a matrix index's possible values must have the same dimension as the corresponding dimension of the matrix, but it may, in general, have arbitrary offset; in practice, however, the corresponding enumeration sets are usually either zero-offset or unit-offset, and are usually of equal offset for both indices of a given matrix. If no definite specification is made in the text, the default action is to assume a matrix's indices to be unit-offset.

The row index for a column vector, and the column index for a row vector, may be omitted, and the default action is to do so.

A.7.5 Explicit listing of matrix elements

The entries in a matrix may be written out explicitly by arranging them into a rectangular array surrounded by large parentheses, whereby rows and columns take on their typographical meaning.

For an example of the listing of matrix elements, see equation (A.54).

A.8 Relativistic mechanics

A.8.1 Lab frames

This thesis does not consider gravitational questions; thus, the spacetime manifold employed is always flat. The specially relativistic notion of a *global inertial frame* is therefore applicable; such a frame is referred to equivalently as a *lab frame* or a *Lorentz frame*.

(The term *frame* itself connotes the dual concept of a *coördinate system* that is *globally valid*: a frame "moves mechanically" as a rigid whole.)

The definite article is sometimes used to single out a frame that has some importance for a particular pratical application. In such cases, the phrase "the lab frame" is used to distinguish this particular Lorentz frame from the MCLF of the particle in question (*see* Section A.8.16). However, both frames are "lab frames".

On the other hand, the CACS of a particle is *not* a Lorentz frame (*see* Section A.8.17).

Quantities evaluated in a lab frame are described by the adjective *lab*.

A.8.2 Lorentz tensors

Rank-zero, rank-one and rank-two Lorentz-covariant quantities are referred to as *Lorentz scalars*, *Lorentz vectors* and *Lorentz tensors* respectively. Lorentz-covariant tensors of rank $r > 2$ are disambiguated from the rank-two term by referring to them explicitly as *rank-r Lorentz tensors*.

Lorentz vectors are also referred to as *four-vectors*, and Lorentz tensors as *four-tensors*; these terms may optionally be prefixed by the adjective "Lorentz" for disambiguation or emphasis. Physical quantities described by four-vectors may generally be described using the prefix *four-*. Antisymmetric Lorentz tensors are sometimes referred to as *six-vectors*, or *bi-vectors*; the three-vector constructed from the time–space components is referred to as the *electric part*, and the three-vector constructed from the space–space components as the *magnetic part*, whether or not the bi-vector describes the electromagnetic field.

A.8.3 Notation for Lorentz quantities

Lorentz-covariant quantities are denoted by symbols from the standard symbol set. In the case where a four-vector and a four-tensor, or a four-tensor and another four-tensor, are closely related to each other (usually, but not always, via a duality transformation), then, in the case of a four-vector and a four-tensor, the latter may be denoted by the same symbol as the former, adorned with a tilde; and in the case of two four-tensors, one may be arbitrarily chosen, and adorned with a tilde.

The symbol is dereferenced by superscripting with the appropriate spacetime indices, which take values from the zero-offset four-dimensional enumeration set, *i.e.*, the values 0, 1, 2 and 3. The value 0, also given the enumeration name t, denotes the time component of spacetime. The values 1, 2 and 3, also given the enumeration names x, y and z respectively, denote the spatial components. Contravariant indices are superscripted; covariant indices are subscripted.

An index denoted by a Latin symbol takes on the values of the unit-offset three-dimensional enumeration set only, *i.e.*, they run from 1 to 3; indices denoted by Greek symbols take on all values of the enumeration set, *i.e.*, run from 0 to 3.

A.8.4 Raising and lowering indices

The indices of a Lorentz quantities are raised and lowered using the metric tensor g (*see* Section A.8.12).

A.8.5 Explicit listing of components

Components of a four-vector are sometimes required to be shown explicitly. In such a case the components of the four-vector are listed in parentheses, separated by commas; for example,

$$(\gamma, \gamma v_x, \gamma v_y, \gamma v_z). \tag{A.41}$$

The spatial part, being a three-vector (*see* Section A.9), may be replaced by any valid three-vector notation; for the example above,

$$(\gamma, \gamma \boldsymbol{v}).$$

In each case the components listed are always the *contravariant* components of the four-vector.

The components of a four-tensor sometimes need to be listed explicitly. Matrix notation is used for such purposes (*see* Section A.7.5).

A.8.6 Outer-products

Lorentz-covariant quantities placed adjacent to each other are considered to represent the *outer-product*, *i.e.*, the simple product of the components; thus,

AB denotes the four-tensor that is the outer-product of the four-vectors A and B:

$$(AB)^{\alpha\beta} \equiv A^\alpha B^\beta. \tag{A.42}$$

A.8.7 Inner-products, dot-products

A nondereferenced four-vector with a central dot either preceding or following it denotes the operation variously known as the *contraction*, *inner-product* or *dot-product* of that four-vector with the quantity lying to the other side of the central dot; for example, given two four-vectors A and B, then

$$A \cdot B \equiv A^\alpha B_\alpha. \tag{A.43}$$

A nondereferenced four-tensor with a central dot preceding (following) it denotes the dot-product of that four-tensor with the quantity lying to the left (right) of the central dot, over the first (second) index of the four-tensor. For example, if A and B are four-vectors and F a four-tensor, then $A \cdot F$ has components

$$(A \cdot F)^\alpha \equiv A^\beta F_\beta{}^\alpha, \tag{A.44}$$

and $F \cdot B$ has components

$$(F \cdot B)^\alpha \equiv F^\alpha{}_\beta B^\beta. \tag{A.45}$$

Four-tensors may be dot-producted on both sides at once; for example,

$$A \cdot F \cdot B \equiv A^\alpha F_\alpha{}^\beta B_\beta, \tag{A.46}$$

and may be "chained" together into longer dot-products:

$$A \cdot F \cdot F \cdot B \equiv A^\alpha F_\alpha{}^\beta F_\beta{}^\mu B_\mu. \tag{A.47}$$

In any of the above dot-products using the central dot, in which all indices are fully contracted (*i.e.*, the result is a Lorentz scalar), parentheses are often used to visually delineate the product; *e.g.*,

$$D(A \cdot F \cdot B) \equiv DA \cdot F \cdot B, \tag{A.48}$$

where the first product is of course an outer-product (simple multiplication).

Parentheses or brackets *must* be used around proper-time derivatives of kinematical quantities where there is a difference between the partial and covariant derivatives (*see* Section A.8.22).

A.8.8 Chained dot-product rings

If a central dot both precedes and follows a product of one or more nondereferenced four-tensors, an inner-product over the first index of the leftmost tensor and the second index of the rightmost tensor is deemed to take place; for example, if F, G and H are four-tensors, then

$$\cdot F \cdot \equiv F^{\alpha}{}_{\alpha}, \tag{A.49}$$

$$\cdot F \cdot G \cdot \equiv F^{\alpha}{}_{\beta} G^{\beta}{}_{\alpha}, \tag{A.50}$$

$$\cdot F \cdot G \cdot H \cdot \equiv F^{\alpha}{}_{\beta} G^{\beta}{}_{\gamma} H^{\gamma}{}_{\alpha}. \tag{A.51}$$

The notation is meant to imply that the factors are conceptually wrapped into a ring, whereby the leading and trailing central dots would be merged into one.

A.8.9 Epsilon products

If A, B, C and D are four-vectors, then the *epsilon product* of them is defined as

$$\varepsilon(A, B, C, D) \equiv \varepsilon_{\alpha\beta\mu\nu} A^{\alpha} B^{\beta} C^{\mu} D^{\nu}. \tag{A.52}$$

If one entry in the $\varepsilon(\,)$ symbol is a four-tensor, and there are only two commas in the symbol, then the four-tensor is deemed to carry the two adjacent indices in the inner-product. Likewise, if there are two four-tensor entries in the $\varepsilon(\,)$ symbol, and only one comma, then the two four-tensors are deemed to carry the four indices. For example, if A and B are four-vectors, and F

and G are four-tensors, then

$$\begin{aligned}\varepsilon(A,B,F) &\equiv \varepsilon_{\alpha\beta\mu\nu}A^\alpha B^\beta F^{\mu\nu},\\ \varepsilon(A,F,B) &\equiv \varepsilon_{\alpha\beta\mu\nu}A^\alpha F^{\beta\mu} B^\nu,\\ \varepsilon(F,A,B) &\equiv \varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta} A^\mu B^\nu,\\ \varepsilon(F,G) &\equiv \varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta} G^{\mu\nu}.\end{aligned}$$

If one of the entries in the $\varepsilon(\,)$ symbol is vacant, then the overall symbol represents a four-vector with the vacant entry as the vector index; for example, $\varepsilon(,B,C,D)$ has components

$$\Big(\varepsilon(,B,C,D)\Big)^\alpha \equiv \varepsilon^\alpha{}_{\beta\mu\nu}B^\beta C^\mu D^\nu.$$

If two entries in the $\varepsilon(\,)$ symbol are vacant, then the overall symbol represents a four-tensor, where the first vacant entry represents the first index, and the second vacant entry the second index; for example, $\varepsilon(,B,,D)$ has components

$$\Big(\varepsilon(,B,,D)\Big)^{\alpha\mu} \equiv \varepsilon^\alpha{}_\beta{}^\mu{}_\nu B^\beta D^\nu.$$

A.8.10 Cross-products

An alternative notation for the epsilon product is that of the *four-cross-product*. It is defined, for four-vectors B, C and D, as

$$B\times C\times D \equiv \varepsilon(,B,C,D). \qquad \text{(A.53)}$$

(This is analogous to the three-vector case; *see* Section A.9.13.)

Either pair of adjacent four-vectors may be replaced by a four-tensor; *e.g.*, if F is a four-tensor, then

$$\begin{aligned}F\times D &\equiv \varepsilon(,F,D),\\ B\times F &\equiv \varepsilon(,B,F).\end{aligned}$$

The fully-contracted quantity $\varepsilon(A, B, C, D)$ may, as with the three-dimensional case, be written by using a combination of the dot and cross products:

$$\varepsilon(A, B, C, D) \equiv A \cdot B \times C \times D.$$

Additionally, the cross-product symbol is defined to be valid for only two four-vector factors, namely,

$$C \times D \equiv \varepsilon(\,,\,, C, D).$$

If the four-vectors C and D are to be replaced by a four-tensor F, the cross symbol that would otherwise appear between C and D may be placed above the symbol F:

$$\overset{\times}{F} \equiv \varepsilon(\,,\,, F).$$

Finally, one may also place a cross above a single three-vector D; this is defined as

$$\overset{\times}{D} \equiv \varepsilon(\,,\,,\,, D).$$

A.8.11 Wedge-products

The *wedge product* of two four-vectors A and B is defined as

$$A \wedge B \equiv A^{[\alpha} B^{\beta]}.$$

A.8.12 Metric tensor

The fully covariant and fully contravariant metric tensors are denoted $g_{\mu\nu}$ and $g^{\mu\nu}$ respectively. They both have the explicit components

$$(g_{\mu\nu}) \equiv (g^{\mu\nu}) \equiv \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{A.54}$$

in a lab frame.

The use of this signature of metric means that raising or lowering a time component index of a quantity does not change the value of the quantity; on the other hand, it does mean that the position of placement of *Latin* indices on the symbols of Lorentz-covariant quantities is important. However, the position of placement of indices on *non-Lorentz-covariant* three-vectors is defined to be in all cases irrelevant, just as it is for mathematically non-Lorentz-covariant three-vectors (*see* Section A.9). For example, with x^α the symbol for the four-position, x^i and x_i are different (indeed, $x^i \equiv -x_i$); on the other hand $v_i \equiv d_t x^i$ requires no distinction, as it is a non-covariant quantity; it may, if convenient, be denoted v^i, but in all cases $v_i \equiv v^i$. This implicit convention is of use, conceptually, when one side of an equation or definition involves (implicitly or explicitly) a non-covariant three-vector, and the other the spatial part of a covariant quantity (see, *e.g.*, the definition of $\boldsymbol{v}$ below). In practice, however, subscripts are generally used on non-covariant quantities in explicit expressions.

A.8.13 Timelike, spacelike, lightlike

If A is a four-vector, then it is *timelike* if $A^2 > 0$; it is *spacelike* if $A^2 < 0$; it is *lightlike* if $A^2 = 0$.

If, in some particular Lorentz frame, $\boldsymbol{A} = \mathbf{0}$, then A is *purely timelike* in that frame; if $A^0 = 0$, then A is *purely spacelike* in that frame.

A.8.14 Four-position of a particle

The Lorentz-covariant *four-position* of a classical particle is denoted z^α.

A.8.15 Three-velocity of a particle

The non-covariant *three-velocity* of a classical particle, as seen in some given lab frame, is denoted $\boldsymbol{v}$, and is defined as

$$\boldsymbol{v} \equiv d_t \boldsymbol{z}, \tag{A.55}$$

where t is the time-coördinate in that frame.

A.8.16 MCLF of a particle

The *momentarily comoving Lorentz frame*, or *MCLF*, of a classical particle at any given instant of time is defined to be the lab frame in which the particle's instantaneous three-velocity $\boldsymbol{v}$ vanishes, and its mechanical energy is greater than zero.

The MCLF is often referred to by the alternative term *instantaneous rest frame*.

Note that the MCLF is a Lorentz frame, and remains so for all time. If the particle is being accelerated, then the MCLF at one moment will not be the MCLF at the next instant; the latter is a new Lorentz frame.

A.8.17 CACS of a particle

The *co-accelerated coördinate system* of a classical particle, or CACS, is a system of coördinates that is co-accelerated with the particle.

Unless the particle is unaccelerated for all time, the CACS does *not* constitute a Lorentz frame.

A.8.18 MCLF and CACS components

Components of quantities evaluated in the MCLF (or, in general, any other Lorentz frame) may be denoted by surrounding the dereferenced quantity by brackets; *e.g.*,

$$C^\alpha|_{\rm MCLF} \equiv [C^\alpha]. \tag{A.56}$$

Components of quantities evaluated in the CACS, on the other hand, are denoted by surrounding the quantity by parentheses, and then dereferencing the resulting symbol; *e.g.*,

$$C^\alpha|_{\rm CACS} \equiv (C)^\alpha. \tag{A.57}$$

These notational distinctions are of vital importance when considering proper-time derivatives; *see* Section A.8.22.

The use of brackets and parentheses to denote the difference between MCLF and CACS components overrides the otherwise general equivalence of these binding symbols specified in Section A.3.8.

We extend the notation to products of Lorentz quantities also: if C and D are kinematical four-vectors, then

$$\begin{aligned}
(C\cdot D) &\equiv (C)\cdot(D), \\
[C\cdot D] &\equiv [C]\cdot[D], \\
(C^{2m}) &\equiv (C)^{2m}, \\
[D^{2m}] &\equiv [D]^{2m}, \\
(C\wedge D) &\equiv (C)\wedge(D), \\
[C\wedge D] &\equiv [C]\wedge[D].
\end{aligned}$$

If the MCLF and CACS components of some particular Lorentz quantity are numerically identical, when referred to an arbitrary lab frame, then either brackets *or* parentheses, or neither, may be used around the Lorentz quantity's symbol, as convenient. If the referred components are *not* identical, however, either brackets or parentheses *must* be used, or else the expression is in error.

A.8.19 Proper time of a particle

The *proper time* for a classical particle is denoted τ, and is defined as the cumulative time measured in the particle's CACS.

Referred to an arbitrary lab frame with coördinates $(t, \boldsymbol{x})$, the differential of τ is given by

$$d\tau \equiv \pm dt\sqrt{1-\boldsymbol{v}^2}, \tag{A.58}$$

where the sign is $+$ $(-)$ if the particle's mechanical energy in the lab frame is positive (negative).

A.8.20 Overdots

An *overdot* on a Lorentz-covariant quantity always denotes d_τ. Refer to Sections A.8.21 and A.8.22 for a description of the various ways the proper-time derivative may be computed.

An overdot on a non-covariant quantity always denotes d_t.

A.8.21 Convective derivative

The proper-time derivative of a quantity that is "external" to the particle in question (*i.e.*, is not an intrinsic quality of the particle itself, but is due to an outside agent; *e.g.*, an externally applied field) is given by the *relativistic convective derivative*,

$$d_\tau E \equiv (U\cdot\partial)E, \tag{A.59}$$

where E is the external quantity in question. Overdots are *not* to be used to denote such a derivative.

A.8.22 Kinematical proper-time derivatives

The proper-time derivative of an intrinsic kinematical property of a particle can be computed in two different ways.

The first way is to simply measure the *time-derivatives of the components* of covariant quantities, in the MCLF of the particle: we [65] refer to this as the *partial proper-time derivative* of the quantity. Using the notation (A.56), we have

$$d_\tau\Big\{[C^\alpha]\Big\} \equiv [\dot{C}^\alpha]. \tag{A.60}$$

The second way to define the proper-time derivative of a kinematical quantity is to compute it *as seen in the CACS*. In other words, we need to compute the proper-time derivative of the covariant quantity, in an abstract way, along the particle's motion, and *then* evaluate its components (if required) in some Lorentz frame: we refer to this as the *covariant proper-time derivative*. Using the notation (A.57), we have

$$\Big\{d_\tau C\Big\}^\alpha \equiv (\dot{C})^\alpha. \tag{A.61}$$

It is shown in Chapter 2 that the connection between the partial and covariant proper-time derivatives, for an arbitrary kinematical four-vector C, is given by

$$(\dot{C})^\alpha = [\dot{C}^\alpha] + U^\alpha(\dot{U}\cdot C). \tag{A.62}$$

A.8.23 Four-velocity of a particle

The *four-velocity* of a classical particle is denoted U^α, and is defined as

$$U^\alpha \equiv \dot{z}^\alpha. \tag{A.63}$$

A.8.24 Mechanical four-momentum

The *mechanical four-momentum* of a classical particle of mass m is denoted p^α, and is defined as

$$p^\alpha \equiv mU^\alpha.$$

A *massless particle* is a particle for which $m = 0$. It is assumed that massless particles have finite mechanical momentum components. Thus, τ and U^α are undefined for massless particles.

A.8.25 Nonrelativistic and pre-relativistic limits

There are two distinct low-velocity limits of special relativity that are recognised throughout this thesis, and they are not equivalent.

We call the *prerelativistic limit* of the motion of a classical particle that limit obtained by only retaining terms up to first order in the three-velocity of the particle or any of its lab-time derivatives, *i.e.*, any of $\boldsymbol{v}$, $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, $\ldots$, but *discarding* any terms jointly proportional to more than one of these quantities. Note carefully that the acceleration, jerk, *etc.* (*i.e.*, $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, *etc.*) *may not* be replaced in any way (such as by using equations of motion) before taking the prerelativistic limit.

The prerelativistic limit only achieves its full power in the CACS of the particle. Under these conditions, the prerelativistic equations of motion, if obtainable by first principles, may be used to uniquely and rigorously obtain the fully covariant equations of motion for the particle.

We call the *nonrelativistic* limit that limit obtained by *replacing* the acceleration, jerk, *etc.* (*i.e.*, $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, *etc.*) by expressions possibly involving $\boldsymbol{v}$ (but no time-derivatives of $\boldsymbol{v}$) wherever they appear in the fully relativistic equations, by using the equations of motion, together with any constitutive equations available, and *then* retaining only terms up to first order in $\boldsymbol{v}$. It should be noted that this is *not* equivalent to the prerelativistic limit defined above. (Most famously, the *Thomas precession* of the prerelativistic limit is converted to a numerical "Thomas factor" in the nonrelativistic limit.) The nonrelativistic limit is less powerful than the prerelativistic limit.

See Section 2.6.12 for a more detailed discussion of these issues.

A.8.26 C, P, T

In any Lorentz frame with coördinates $(t, \boldsymbol{x})$, the operation

$$\boldsymbol{x} \rightarrow -\boldsymbol{x} \tag{A.64}$$

is referred to as the *parity* operation, and is denoted P. The operation

$$t \rightarrow -t \tag{A.65}$$

is referred to as the *time reversal* operation, and is denoted T.

The operation of exchanging particles with antiparticles, and vice versa, is referred to as the *particle–antiparticle conjugation operation*, and is denoted C.

In classical physics, particle–antiparticle conjugation is carried out via the operation

$$\tau \rightarrow -\tau \tag{A.66}$$

for all particles. Thus, in classical physics, the CPT theorem is axiomatic.

A.9 Euclidean three-space

We deem the prefixes "*three-*" and "*three-space-*" to be appropriate choices for the prefix G- of Section A.3.18 when discussing manifestly covariant treatments of three-dimensional Euclidean space. The choice of prefix in any particular situation is arbitrary, and is generally made on the basis of nomenclatorial clarity.

A.9.1 Three-tensors

Rank-zero, rank-one and rank-two three-space tensors are referred to as *three-scalars*, *three-vectors* and *three-tensors* respectively. Three-covariant tensors of rank $r > 2$ are disambiguated from the rank-two term by referring to them explicitly as *rank-r three-tensors*. Three-tensors are also referred to as *three-dyads*.

A.9.2 Notation for three-space quantities

Three-scalars are denoted by symbols from the standard symbol set. Three-vectors and three-dyads are denoted by *boldfacing* a symbol from the standard symbol set.

A three-vector or three-dyad symbol is dereferenced by removing the boldness of the face and subscripting or superscripting the resulting symbol with the appropriate index or indices, which take values from the unit-offset three-dimensional enumeration set, *i.e.*, the values 1, 2 and 3. These enumerations are also given the enumeration names x, y and z respectively.

If a dereferenced three-vector is a non-Lorentz-covariant quantity, then the choice of whether a subscript or superscript index is used is *arbitrary*; the result is deemed to be in all cases the same. If the three-vector is, on the other hand, the three-vector part of a Lorentz four-vector, then the choice of subscript or superscript will select the covariant or contravariant version respectively of the three-vector part of the given four-vector.

Indices for three-vectors are always Latin elements of the standard symbol set, never Greek, and are preferentially taken from the symbols i, j, k, l, m and n before any others.

A.9.3 The null three-vector

The *null three-vector*, $(0, 0, 0)$, is always denoted $\mathbf{0}$.

A.9.4 The three-gradient operator

The symbol ∇ denotes the three-vector derivative operator, and is defined to have the explicit components

$$\nabla \equiv (\partial_1, \partial_2, \partial_3) \equiv (\partial_x, \partial_y, \partial_z)\,. \qquad \text{(A.67)}$$

The symbol ∂, being a special symbol, does not have a standard boldface three-vector conversion. In any case, the boldfaced typographical symbol $\boldsymbol{\partial}$ is hereby explicitly decommissioned.

A.9.5 Explicit listing of components

Components of a three-vector are sometimes required to be shown explicitly. In such a case the components of the three-vector are listed in parentheses, separated by commas; for example,

$$(v_x, v_y, v_z). \tag{A.68}$$

The components of a three-dyad sometimes need to be listed explicitly. Matrix notation is used for such purposes.

A.9.6 Mixed notation

Different choices of three-vector notation may be used for different parts of an expression or equation, if convenient and unambiguous. In particular, a different choice of notation may be used for either side of an equation, but *only* if the equation is a scalar or vector one. Thus,

$$\boldsymbol{A}\cdot\boldsymbol{B} = C_i D_i \tag{A.69}$$

is a valid (scalar) equation, as are

$$\boldsymbol{U} = (\gamma v_x, \gamma v_y, \gamma v_z) \tag{A.70}$$

and

$$\boldsymbol{A}\times\boldsymbol{B} = \varepsilon_{ijk} A_j B_k \tag{A.71}$$

(vector equations—the unpaired index i in (A.71) automatically taken to be the implicit index of the left hand side), but equations such as

$$A_{ij} = B_i C_j \tag{A.72}$$

(three-dyad equations) may not be rewritten in mixed notation.

A.9.7 Uniqueness of symbols

No boldfaced symbol in this thesis is used in the same section for both a three-vector and a three-dyad.

No boldfaced symbol in this thesis is used in the same section for two different three-vectors unless the three-vectors are identical. This implies that if some given three-vector is the three-vector part of a Lorentz four-vector, and another three-vector is not, then they may not be assigned the same symbol. This further implies that a four-vector may not be denoted by the same symbol as an existing three-vector unless the three-vector part of the four-vector is identical to the existing three-vector, and vice versa. For example, since the non-covariant three-velocity is denoted by the symbol $\boldsymbol{v}$, the four-velocity $(\gamma, \gamma\boldsymbol{v})$ could *not* be denoted v^α, since the three-vector part of this, equal to $\gamma\boldsymbol{v}$, would implicitly be denoted by $\boldsymbol{v}$, which is an absurdity.

When a three-vector denoted by boldface is actually the three-vector part of a Lorentz four-vector, it is always the *contravariant* three-vector that is denoted. Conversely, the use of the boldface of a symbol that was previously defined as a four-vector denotes, without need for explicit comment, the contravariant three-vector part of that four-vector. Thus, if C^α is some arbitrary four-vector, then

$$\boldsymbol{C} \equiv (C^1, C^2, C^3)$$

implicitly, so that once C^α is defined, the quantity $\boldsymbol{C}$ may be used without explicit comment.

A.9.8 Parentheses and brackets

The placing of brackets or parentheses around the boldface of a four-vector symbol disambiguates the MCLF and CACS evaluations of the three-vector part of that four-vector (*see* Section A.8.18).

A.9.9 Magnitude of a three-vector

The *magnitude* of a three-vector $\boldsymbol{A}$ is defined as

$$\sqrt{\boldsymbol{A}^2}. \tag{A.73}$$

If, and only if, a three-vector is *not* the three-vector part of any Lorentz four-vector, then the magnitude of the three-vector may be denoted by the symbol of the three-vector with the boldface removed, without explicit comment. For example, if $\boldsymbol{A}$ is a three-vector, but *not* part of any four-vector A^μ, then its magnitude (A.73) may be referred to as simply A, without need for explicit comment.

A.9.10 Outer-products

Three-vectors placed adjacent to each other are considered to represent the *outer-product*, *i.e.*, the simple product of the components; thus, $\boldsymbol{AB}$ denotes the three-dyad that is the outer-product of the three-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$(\boldsymbol{AB})_{ij} \equiv A_i B_j. \tag{A.74}$$

A.9.11 Dot-products

The *dot-product* of two three-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A}\cdot\boldsymbol{B} \equiv A_i B_i.$$

A nondereferenced three-dyad with a central dot preceding (following) it denotes the dot-product of that three-dyad with the quantity lying to the left (right) of the central dot, over the first (second) index of the three-dyad. For example, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are three-vectors and $\boldsymbol{F}$ a three-dyad, then $(\boldsymbol{A}\cdot\boldsymbol{F})$ has components

$$(\boldsymbol{A}\cdot\boldsymbol{F})_i \equiv A_j F_{ji}, \tag{A.75}$$

and $(\boldsymbol{F}\cdot\boldsymbol{B})$ has components

$$(\boldsymbol{F}\cdot\boldsymbol{B})_i \equiv F_{ij}B_j. \tag{A.76}$$

Three-dyads may be dot-producted on both sides at once; for example,

$$(\boldsymbol{A}\cdot\boldsymbol{F}\cdot\boldsymbol{B}) \equiv A_iF_{ij}B_j, \tag{A.77}$$

and may be "chained" together into longer inner-products:

$$(\boldsymbol{A}\cdot\boldsymbol{F}\cdot\boldsymbol{F}\cdot\boldsymbol{B}) \equiv A_iF_{ij}F_{jk}B_k. \tag{A.78}$$

A.9.12 Epsilon products

For any three-vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, the *epsilon product* is defined as

$$\varepsilon(\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}) \equiv \varepsilon_{ijk}A_iB_jC_k. \tag{A.79}$$

As with the case with four-vectors, three-dyads may be used in the $\varepsilon(,,)$ notation: they cover two adjacent positions; *e.g.*, if $\boldsymbol{F}$ is a three-dyad, then

$$\begin{aligned}
\varepsilon(\boldsymbol{A},\boldsymbol{F}) &\equiv \varepsilon_{ijk}A_iF_{jk},\\
\varepsilon(\boldsymbol{F},\boldsymbol{C}) &\equiv \varepsilon_{ijk}F_{ij}C_k,\\
\Big(\varepsilon(,\boldsymbol{F})\Big)_i &\equiv \varepsilon_{ijk}F_{jk},\\
\Big(\varepsilon(\boldsymbol{F},)\Big)_k &\equiv \varepsilon_{ijk}F_{ij}.
\end{aligned}$$

One or two of the entries in the $\varepsilon(,,)$ symbol may be vacant; the symbol then represents a three-vector, or a three-dyad, respectively:

$$\begin{aligned}
\Big(\varepsilon(,\boldsymbol{B},\boldsymbol{C})\Big)_i &\equiv \varepsilon_{ijk}B_jC_k,\\
\Big(\varepsilon(,,\boldsymbol{C})\Big)_{ij} &\equiv \varepsilon_{ijk}C_k.
\end{aligned}$$

A.9.13 Cross-products

An alternative notation for the epsilon product is that of the *three-cross-product*. It is defined, for three-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, as

$$\boldsymbol{A}\times\boldsymbol{B} \equiv \varepsilon(, \boldsymbol{A}, \boldsymbol{B}).$$

Appendix B

Supplementary Identities

B.1 Introduction

In this appendix, we collect together various identities used throughout this thesis, for convenient reference. The notation and conventions used are explained in Appendix A.

B.2 The electromagnetic field

B.2.1 The four-potential

The fundamental physical object characterising the electromagnetic field is the massless four-vector field $A(x)$, which is, for the purposes of this thesis, treated as a classical field, and referred to as the *electromagnetic four-potential*.

$A(x)$ is not physically observable in its own right.

B.2.2 The field strength tensor

The six-vector $F(x)$ is referred to as the *electromagnetic field strength tensor*, and is obtained from $A(x)$ by means of the definition

$$F \equiv \partial \wedge A. \tag{B.1}$$

B.2.3 The homogeneous Maxwell equations

From the definition (B.1), one immediately finds the *homogeneous Maxwell equations*,

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} \equiv 0. \tag{B.2}$$

Note that magnetic charges ("monopoles") cannot exist if the four-potential $A(x)$ is assumed to be fundamental.

B.2.4 The dual field strength tensor

The *dual electromagnetic field strength tensor*, $\widetilde{F}(x)$, is obtained from $F(x)$ according to

$$\widetilde{F} \equiv -\frac{1}{2}\overset{\times}{F}, \tag{B.3}$$

where in (B.3) we employ the four-cross-product notation defined in Section A.8.10. From (B.3), one can show that the reverse transformation is

$$F \equiv \frac{1}{2}\overset{\times}{\widetilde{F}}. \tag{B.4}$$

B.2.5 The electromagnetic duality transformation

The minus sign in (B.3) might seem misplaced to some readers. It can be traced back to the sign convention chosen for the alternating function in Section A.5.4. The sign of (B.3) has, in fact, been chosen so that the explicit electric and magnetic parts of $\widetilde{F}$ are *always* related to those of F by means of the *electromagnetic duality transformation*:

$$\begin{aligned} \boldsymbol{E} &\longrightarrow \boldsymbol{B}, \\ \boldsymbol{B} &\longrightarrow -\boldsymbol{E}. \end{aligned} \tag{B.5}$$

B.2.6 The dual homogeneous Maxwell equation

In terms of $\widetilde{F}$, the homogeneous Maxwell equations (B.6) can be rewritten

$$\partial \cdot \widetilde{F} = 0. \tag{B.6}$$

B.2.7 The inhomogeneous Maxwell equation

The *electromagnetic source current* four-vector, $J(x)$, is defined by

$$J \equiv \partial \cdot F. \tag{B.7}$$

Equation (B.7) is referred to as the *inhomogeneous Maxwell equation*.

The antisymmetry of F, from (B.1), then shows that $J(x)$ is a conserved current, *i.e.*,

$$\partial \cdot J = 0. \tag{B.8}$$

Physical objects are often *ascribed* an electromagnetic source current *a priori*.

B.2.8 The dual inhomogeneous Maxwell equations

In terms of the dual electromagnetic field strength tensor, $\widetilde{F}$, equation (B.7) can be written

$$\partial_\alpha \widetilde{F}_{\beta\gamma} + \partial_\beta \widetilde{F}_{\gamma\alpha} + \partial_\gamma \widetilde{F}_{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma\mu} J^\mu. \tag{B.9}$$

B.2.9 Explicit electromagnetic components

For non-manifestly-covariant analyses, the explicit four-vector components

$$A^\mu \equiv (\phi, \boldsymbol{A}), \tag{B.10}$$

$$J^\mu \equiv (\rho, \boldsymbol{J}), \tag{B.11}$$

and the explicit four-tensor components

$$F_{\alpha\beta} \equiv \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{B.12}$$

$$\widetilde{F}_{\alpha\beta} \equiv \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z & -E_y \\ -B_y & -E_z & 0 & E_x \\ -B_z & E_y & -E_x & 0 \end{pmatrix}, \tag{B.13}$$

are employed; in other words, $\boldsymbol{E}$ and $\boldsymbol{B}$ are defined as

$$\begin{aligned} \boldsymbol{E} &\equiv -\nabla\phi - \partial_t \boldsymbol{A}, \\ \boldsymbol{B} &\equiv \nabla\times\boldsymbol{A}. \end{aligned} \tag{B.14}$$

B.2.10 Explicit Maxwell equations

In terms of the explicit quantities ρ, $\boldsymbol{J}$, $\boldsymbol{E}$ and $\boldsymbol{B}$, the Maxwell equations (B.6) and (B.7) become

$$\nabla\cdot\boldsymbol{B} \equiv 0, \tag{B.15}$$

$$\nabla\times\boldsymbol{E} + \partial_t\boldsymbol{B} \equiv \boldsymbol{0}, \tag{B.16}$$

$$\nabla\cdot\boldsymbol{E} \equiv \rho, \tag{B.17}$$

$$\nabla\times\boldsymbol{B} - \partial_t\boldsymbol{E} \equiv \boldsymbol{J}, \tag{B.18}$$

while the source current conservation equation (B.8) becomes

$$\partial_t\rho + \nabla\cdot\boldsymbol{J} \equiv 0. \tag{B.19}$$

B.2.11 Quadratic field identities

By multiplying out the matrices (B.12) and (B.13), after raising one index of both using the metric, one may verify that

$$\cdot F\cdot F\cdot \equiv 2(\boldsymbol{E}^2 - \boldsymbol{B}^2), \tag{B.20}$$

$$\cdot \widetilde{F}\cdot \widetilde{F}\cdot \equiv 2(\boldsymbol{B}^2 - \boldsymbol{E}^2), \tag{B.21}$$

$$\cdot F\cdot \widetilde{F}\cdot \equiv 4(\boldsymbol{E}\cdot\boldsymbol{B}), \tag{B.22}$$

$$\cdot \widetilde{F}\cdot F\cdot \equiv 4(\boldsymbol{E}\cdot\boldsymbol{B}), \tag{B.23}$$

$$F\cdot \widetilde{F} \equiv \frac{1}{4}(\cdot F\cdot \widetilde{F}\cdot)g, \tag{B.24}$$

$$\widetilde{F}\cdot F \equiv \frac{1}{4}(\cdot F\cdot \widetilde{F}\cdot)g, \tag{B.25}$$

$$F\cdot F - \widetilde{F}\cdot \widetilde{F} \equiv \frac{1}{2}(\cdot F\cdot F\cdot)g. \tag{B.26}$$

B.2.12 Other field identities

The following three identities are proved in Appendix C:

$$\partial(U\cdot A) - (U\cdot\partial)A = F\cdot U, \tag{B.27}$$

$$(U\cdot\partial)F\cdot\Sigma + \partial(\Sigma\cdot F\cdot U) = (\Sigma\cdot\partial)F\cdot U, \tag{B.28}$$

$$(U\cdot\partial)\widetilde{F}\cdot\Sigma + \partial(\Sigma\cdot\widetilde{F}\cdot U) = (\Sigma\cdot\partial)\widetilde{F}\cdot U + \Sigma\times J\times U. \tag{B.29}$$

B.3 Three-vectors

If ψ is a three-scalar, and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are three-vectors, all of which are c-number quantities, then [113]

$$\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c} \equiv \boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a} \equiv \boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}, \tag{B.30}$$

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) \equiv (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}, \tag{B.31}$$

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) \equiv (\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{c}), \tag{B.32}$$

$$\nabla\times\nabla\psi \equiv \mathbf{0}, \tag{B.33}$$

$$\nabla\cdot(\nabla\times\boldsymbol{a}) \equiv 0, \tag{B.34}$$

$$\nabla\times(\nabla\times\boldsymbol{a}) \equiv \nabla(\nabla\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}, \tag{B.35}$$

$$\nabla\cdot(\psi\boldsymbol{a}) \equiv \boldsymbol{a}\cdot\nabla\psi + \psi\nabla\cdot\boldsymbol{a}, \tag{B.36}$$

$$\nabla\times(\psi\boldsymbol{a}) \equiv (\nabla\psi)\times\boldsymbol{a} + \psi\nabla\times\boldsymbol{a}, \tag{B.37}$$

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) \equiv (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}), \tag{B.38}$$

$$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) \equiv \boldsymbol{b}\cdot\nabla\times\boldsymbol{a} - \boldsymbol{a}\cdot\nabla\times\boldsymbol{b}, \tag{B.39}$$

$$\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) \equiv \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a}) + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}. \tag{B.40}$$

Also,

$$(\boldsymbol{a}\cdot\nabla)\boldsymbol{x} \equiv \boldsymbol{a}. \tag{B.41}$$

B.4 Alternating functions

The three- and four-dimensional alternating functions are connected by

$$\varepsilon_{ijk} \equiv \varepsilon_{0ijk} \equiv \varepsilon_{k0ji} \equiv \varepsilon_{jk0i} \equiv \varepsilon_{kji0}, \tag{B.42}$$

where

$$\varepsilon_{123} \equiv \varepsilon_{0123} \equiv +1. \tag{B.43}$$

Products of two four-dimensional alternating functions, contracted over one, two, three and four indices, are respectively given by

$$\begin{aligned}
\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\tau\mu\nu} &\equiv -\delta^\beta_\tau\delta^\gamma_\mu\delta^\delta_\nu + \delta^\beta_\mu\delta^\gamma_\tau\delta^\delta_\nu - \delta^\beta_\mu\delta^\gamma_\nu\delta^\delta_\tau + \delta^\beta_\nu\delta^\gamma_\mu\delta^\delta_\tau - \delta^\beta_\nu\delta^\gamma_\tau\delta^\delta_\mu + \delta^\beta_\tau\delta^\gamma_\nu\delta^\delta_\mu,\\
\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\mu\nu} &\equiv -2\delta^\gamma_\mu\delta^\delta_\nu + 2\delta^\gamma_\nu\delta^\delta_\mu,\\
\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\nu} &\equiv -6\delta^\delta_\nu,\\
\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} &\equiv -24.
\end{aligned} \tag{B.44}$$

B.5 Four-vector cross-product

The four-cross-product $A \times B \times C$ has explicit components

$$A \times B \times C \equiv \Big(\boldsymbol{A} \cdot \boldsymbol{B} \times \boldsymbol{C},\ A^0 \boldsymbol{B} \times \boldsymbol{C} + B^0 \boldsymbol{C} \times \boldsymbol{A} + C^0 \boldsymbol{A} \times \boldsymbol{B} \Big). \tag{B.45}$$

Relation (B.45) illustrates explicitly the invariance of the four-cross-product under a cyclic permutation of its three four-vector entries:

$$A \times B \times C \equiv B \times C \times A \equiv C \times A \times B. \tag{B.46}$$

B.6 Radiation reaction gradients

The following identities are of use when computing the gradient terms in the radiation reaction calculations of Chapter 6. (The three-vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are "external" quantities—such as $\dot{\boldsymbol{v}}$—that are independent of $\boldsymbol{r}$.) Note that, since the resulting quantities are often to be integrated over all of $\boldsymbol{r}_d$–$\boldsymbol{r}_s$ space, symmetry may be used to eliminate many terms from the calculations.

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \nabla) r_d^m &= m r_d^{m-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}), \\
(\boldsymbol{\sigma} \cdot \nabla) r_d^m \boldsymbol{n}_d &= r_d^{m-1} \Big\{ \boldsymbol{\sigma} + (m-1)(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \Big\}, \\
(\boldsymbol{\sigma} \cdot \nabla) r_d^m (\boldsymbol{n}_d \cdot \boldsymbol{A})^p &= r_d^{m-1} \Big\{ p(\boldsymbol{A} \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \boldsymbol{A})^{p-1} \\
&\qquad + (m-p)(\boldsymbol{n}_d \cdot \boldsymbol{A})^p (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \Big\}, \\
(\boldsymbol{\sigma} \cdot \nabla) r_d^m (\boldsymbol{n}_d \cdot \boldsymbol{A})^p \boldsymbol{n}_d &= r_d^{m-1} \Big\{ p(\boldsymbol{A} \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \boldsymbol{A})^{p-1} \boldsymbol{n}_d + (\boldsymbol{n}_d \cdot \boldsymbol{A})^p \boldsymbol{\sigma} \\
&\qquad + (m-p-1)(\boldsymbol{n}_d \cdot \boldsymbol{A})^p (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \Big\}, \\
(\boldsymbol{\sigma} \cdot \nabla) r_d^m (\boldsymbol{n}_d \cdot \boldsymbol{A})(\boldsymbol{n}_d \cdot \boldsymbol{B}) \boldsymbol{n}_d &= r_d^{m-1} \Big\{ (\boldsymbol{n}_d \cdot \boldsymbol{A})(\boldsymbol{n}_d \cdot \boldsymbol{B}) \boldsymbol{\sigma} \\
&\qquad + (\boldsymbol{A} \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \boldsymbol{B}) \boldsymbol{n}_d \\
&\qquad + (\boldsymbol{B} \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \boldsymbol{A}) \boldsymbol{n}_d \\
&\qquad + (m-3)(\boldsymbol{n}_d \cdot \boldsymbol{A})(\boldsymbol{n}_d \cdot \boldsymbol{B})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \Big\}, \\
(\boldsymbol{\sigma} \cdot \nabla) r_d^m r_s (\boldsymbol{n}_s \cdot \boldsymbol{A}) &= r_d^m (\boldsymbol{A} \cdot \boldsymbol{\sigma}) + m r_d^{m-1} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \boldsymbol{A}),
\end{aligned}$$

$$
\begin{aligned}
(\boldsymbol{\sigma}\cdot\nabla) r_d^m r_s(\boldsymbol{n}_s\cdot\boldsymbol{A})\boldsymbol{n}_d &= r_d^{m-1} r_s\Big\{(\boldsymbol{n}_s\cdot\boldsymbol{A})\boldsymbol{\sigma} \\
&\qquad\qquad + (m-1)(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A})\boldsymbol{n}_d\Big\} \\
&\quad + r_d^m(\boldsymbol{A}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d, \\
(\boldsymbol{\sigma}\cdot\nabla) r_s(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B}) &= r_d^{-1} r_s\Big\{(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B}) \\
&\qquad\qquad - (\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B})\Big\} \\
&\quad + (\boldsymbol{B}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\boldsymbol{A}), \\
(\boldsymbol{\sigma}\cdot\nabla) r_d^m r_s(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d &= r_d^{m-1} r_s\Big\{(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d \\
&\qquad\qquad + (\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{\sigma} \\
&\qquad\qquad + (m-2)(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d\Big\} \\
&\quad + r_d^m(\boldsymbol{B}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\boldsymbol{A})\boldsymbol{n}_d, \\
(\boldsymbol{\sigma}\cdot\nabla) r_s^2(\boldsymbol{n}_s\cdot\boldsymbol{A})^2 &= 2r_s(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A}), \\
(\boldsymbol{\sigma}\cdot\nabla) r_d^{-1} r_s^2(\boldsymbol{n}_s\cdot\boldsymbol{A})^2 &= 2r_d^{-1} r_s(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A}) \\
&\quad - r_d^{-2} r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A})^2, \\
(\boldsymbol{\sigma}\cdot\nabla) r_d^{-1} r_s^2(\boldsymbol{n}_s\cdot\boldsymbol{A})^2\boldsymbol{n}_d &= r_d^{-2} r_s\Big\{(\boldsymbol{n}_s\cdot\boldsymbol{A})^2\boldsymbol{\sigma} - 2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A})^2\boldsymbol{n}_d\Big\} \\
&\quad + 2r_d^{-1} r_s(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{A})\boldsymbol{n}_d, \\
(\boldsymbol{\sigma}\cdot\nabla) r_s(\boldsymbol{n}_d\cdot\boldsymbol{A})^2(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d &= r_d^{-1} r_s\Big\{(\boldsymbol{n}_d\cdot\boldsymbol{A})^2(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{\sigma} \\
&\qquad\qquad + 2(\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d \\
&\qquad\qquad - 3(\boldsymbol{n}_d\cdot\boldsymbol{A})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d\Big\} \\
&\quad + (\boldsymbol{B}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\boldsymbol{A})^2\boldsymbol{n}_d, \\
(\boldsymbol{\sigma}\cdot\nabla) r_d^{-1} r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})^2\boldsymbol{n}_d &= r_d^{-2} r_s^2\Big\{(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})^2\boldsymbol{\sigma} \\
&\qquad\qquad + (\boldsymbol{A}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\boldsymbol{B})^2\boldsymbol{n}_d \\
&\qquad\qquad - 3(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\Big\} \\
&\quad + 2r_d^{-1} r_s(\boldsymbol{B}\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\boldsymbol{A})(\boldsymbol{n}_s\cdot\boldsymbol{B})\boldsymbol{n}_d. \qquad \text{(B.47)}
\end{aligned}
$$

Appendix C

Supplementary Proofs

C.1 Introduction

In this appendix, we collect together various proofs whose explicit inclusion in the body of this thesis was considered unwarranted, but whose inclusion as reference material may be useful to some readers.

C.2 Mechanical field excesses

In this section, we derive the Coulomb-gauge expressions for the excess in the field mechanical momentum and energy caused by an electric charge being brought into the vicinity of the other electric sources in the Universe, in the nonrelativistic limit.

C.2.1 Electric charge mechanical energy excess

(This proof is taken directly from Section 1.11 of Jackson's textbook [113].)

The electromagnetic field mechanical energy is given by the following integral over all space:

$$W_{\text{field}} = \frac{1}{2} \int d^3r \, \boldsymbol{E}^2(\boldsymbol{r}). \tag{C.1}$$

For two electric charges q_1 and q_2, at positions $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$, their Coulomb fields yield

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q_1(\boldsymbol{r}-\boldsymbol{z}_1)}{4\pi\,|\boldsymbol{r}-\boldsymbol{z}_1|^3} + \frac{q_2(\boldsymbol{r}-\boldsymbol{z}_2)}{4\pi\,|\boldsymbol{r}-\boldsymbol{z}_2|^3},$$

so the resulting field energy (C.1) is

$$\begin{aligned} W_{\text{field}} \;=\; & \frac{1}{2}\frac{q_1^2}{(4\pi)^2}\int d^3r\,\frac{1}{|\boldsymbol{r}-\boldsymbol{z}_1|^4} + \frac{1}{2}\frac{q_2^2}{(4\pi)^2}\int d^3r\,\frac{1}{|\boldsymbol{r}-\boldsymbol{z}_2|^4} \\ & + \frac{q_1q_2}{(4\pi)^2}\int d^3r\,\frac{(\boldsymbol{r}-\boldsymbol{z}_1)\cdot(\boldsymbol{r}-\boldsymbol{z}_2)}{|\boldsymbol{r}-\boldsymbol{z}_1|^3\,|\boldsymbol{r}-\boldsymbol{z}_2|^3}. \end{aligned}$$

The first two terms are the field mechanical energy expressions for the charges alone, and are independent of the relative separation of the charges. The *excess* field mechanical energy is therefore given by the last term. A change to the integration variable

$$\boldsymbol{\rho} \equiv \frac{\boldsymbol{r}-\boldsymbol{z}_1}{|\boldsymbol{z}_1-\boldsymbol{z}_2|}$$

yields

$$W_{\text{excess}} = \frac{q_1q_2}{(4\pi)^2\,|\boldsymbol{z}_1-\boldsymbol{z}_2|}\int d^3\rho\,\frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\boldsymbol{n})}{\rho^3\,|\boldsymbol{\rho}+\boldsymbol{n}|^3},$$

where

$$\boldsymbol{n} \equiv \frac{\boldsymbol{z}_1-\boldsymbol{z}_2}{|\boldsymbol{z}_1-\boldsymbol{z}_2|}.$$

Using the identity

$$\frac{\boldsymbol{\rho}+\boldsymbol{n}}{|\boldsymbol{\rho}+\boldsymbol{n}|^3} \equiv -\nabla_\rho\,\frac{1}{|\boldsymbol{\rho}+\boldsymbol{n}|}, \tag{C.2}$$

one finds

$$\int d^3\rho\,\frac{\boldsymbol{\rho}\cdot(\boldsymbol{\rho}+\boldsymbol{n})}{\rho^3\,|\boldsymbol{\rho}+\boldsymbol{n}|^3} = 4\pi, \tag{C.3}$$

and hence

$$W_{\text{excess}} = \frac{q_1q_2}{4\pi\,|\boldsymbol{z}_1-\boldsymbol{z}_2|}. \tag{C.4}$$

Now, since the Coulomb-gauge potential φ generated by charge 1 or 2 at the position of charge 2 or 1 is given by (*see*, *e.g.*, [113, Ch. 1])

$$\begin{aligned}\varphi_1(\boldsymbol{z}_2) &= \frac{q_1}{4\pi\,|\boldsymbol{z}_1 - \boldsymbol{z}_2|},\\ \varphi_2(\boldsymbol{z}_1) &= \frac{q_2}{4\pi\,|\boldsymbol{z}_1 - \boldsymbol{z}_2|},\end{aligned}$$

we find that the excess field mechanical energy (C.4) can be written

$$W_{\text{excess}} = q_2\varphi_1(\boldsymbol{z}_2) = q_1\varphi_2(\boldsymbol{z}_1). \tag{C.5}$$

The linearity of Maxwell's equations, and the quadraticity of the mechanical field energy expression (C.1), then shows that, for any given electric charge q, the excess mechanical field energy caused by the bringing of this charge into any *arbitrary* distribution of electromagnetic scalar potential $\varphi(\boldsymbol{r})$ is given by

$$W_{\text{excess}} = q\varphi(\boldsymbol{z}),$$

where $\boldsymbol{z}$ is the position of the charge q.

C.2.2 Electric charge mechanical momentum excess

The electromagnetic field mechanical momentum is given by the following integral over all space:

$$\boldsymbol{p}_{\text{field}} = \int d^3r\,\boldsymbol{E}(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r}). \tag{C.6}$$

Let us consider placing an electric charge, at rest, amidst some arbitrary magnetic field distribution $\boldsymbol{B}(\boldsymbol{r})$. The Coulomb electric field of the charge, together with the magnetic field distribution $\boldsymbol{B}(\boldsymbol{r})$, will then yield a contribution to (C.6). The Coulomb field of the charge is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{q(\boldsymbol{r}-\boldsymbol{z})}{4\pi\,|\boldsymbol{r}-\boldsymbol{z}|^3},$$

where $\boldsymbol{z}$ is the position of the charge. The magnetic field $\boldsymbol{B}(\boldsymbol{r})$ is itself given by [113, Sec. 5.3]

$$\boldsymbol{B}(\boldsymbol{r}) = \int d^3r' \, \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{4\pi \left|\boldsymbol{r} - \boldsymbol{r}'\right|^3},$$

where $\boldsymbol{J}(\boldsymbol{r}')$ is the electric current distribution generating the magnetic field $\boldsymbol{B}(\boldsymbol{r})$. Thus

$$\boldsymbol{p}_{\text{field}} = \frac{q}{(4\pi)^2} \int d^3r \int d^3r' \, \frac{(\boldsymbol{r} - \boldsymbol{z}) \times [\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')]}{\left|\boldsymbol{r} - \boldsymbol{z}\right|^3 \left|\boldsymbol{r} - \boldsymbol{r}'\right|^3}$$

Now noting that, in the Coulomb gauge [113, Sec. 5.4],

$$\boldsymbol{A}(\boldsymbol{r}) = \int d^3r' \, \frac{\boldsymbol{J}(\boldsymbol{r}')}{4\pi \left|\boldsymbol{r} - \boldsymbol{r}'\right|},$$

and again using the results (C.2) and (C.3), one finds, upon integration,

$$\boldsymbol{p}_{\text{excess}} = q\boldsymbol{A}(\boldsymbol{z}). \tag{C.7}$$

C.3 Electromagnetic field identities

The proofs of equations (B.27), (B.28) and (B.29) proceed as follows:

$$\begin{aligned} \partial(U \cdot A) - (U \cdot \partial)A &= \partial^\alpha (U^\beta A_\beta) - (U^\beta \partial_\beta) A^\alpha \\ &= U^\beta \left\{ \partial^\alpha A_\beta - \partial_\beta A^\alpha \right\} \\ &= U^\beta F^\alpha{}_\beta \\ &= F^\alpha{}_\beta U^\beta \\ &= F \cdot U; \end{aligned}$$

hence,

$$\partial(U \cdot A) - (U \cdot \partial)A = F \cdot U;$$

this is equation (B.27).

$$(U \cdot \partial) F \cdot \Sigma + \partial(\Sigma \cdot F \cdot U) = U^\gamma \partial_\gamma F^\alpha{}_\beta \Sigma^\beta + \partial^\alpha \Sigma^\beta F_{\beta\gamma} U^\gamma$$

$$
\begin{aligned}
&= U^\gamma \Sigma^\beta \Big\{ \partial_\gamma F^\alpha{}_\beta + \partial^\alpha F_{\beta\gamma} \Big\} \\
&= U^\gamma \Sigma^\beta \Big\{ -\partial_\beta F_\gamma{}^\alpha \Big\} \\
&= -U_\gamma (\Sigma \cdot \partial) F^{\gamma\alpha} \\
&= +U_\gamma (\Sigma \cdot \partial) F^{\alpha\gamma} \\
&= (\Sigma \cdot \partial) F \cdot U;
\end{aligned}
$$

hence,

$$
(U \cdot \partial) F \cdot \Sigma + \partial(\Sigma \cdot F \cdot U) = (\Sigma \cdot \partial) F \cdot U;
$$

this is equation (B.28).

$$
\begin{aligned}
(U \cdot \partial)\widetilde{F} \cdot \Sigma + \partial(\Sigma \cdot \widetilde{F} \cdot U) &= U^\gamma \partial_\gamma \widetilde{F}^\alpha{}_\beta \Sigma^\beta + \partial^\alpha \Sigma^\beta \widetilde{F}_{\beta\gamma} U^\gamma \\
&= U^\gamma \Sigma^\beta \Big\{ \partial_\gamma \widetilde{F}^\alpha{}_\beta + \partial^\alpha \widetilde{F}_{\beta\gamma} \Big\} \\
&= U^\gamma \Sigma^\beta \Big\{ -\partial_\beta \widetilde{F}_\gamma{}^\alpha - \varepsilon^\alpha{}_{\beta\gamma\mu} J^\mu \Big\} \\
&= -U_\gamma (\Sigma \cdot \partial) F^{\gamma\alpha} - \varepsilon^\alpha{}_{\beta\gamma\mu} \Sigma^\beta U^\gamma J^\mu \\
&= +U_\gamma (\Sigma \cdot \partial) F^{\alpha\gamma} + \varepsilon^\alpha{}_{\beta\gamma\mu} \Sigma^\beta J^\gamma U^\mu \\
&= (\Sigma \cdot \partial) F \cdot U + \Sigma \times J \times U;
\end{aligned}
$$

hence,

$$
(U \cdot \partial)\widetilde{F} \cdot \Sigma + \partial(\Sigma \cdot \widetilde{F} \cdot U) = (\Sigma \cdot \partial)\widetilde{F} \cdot U + \Sigma \times J \times U;
$$

this is equation (B.29).

Appendix D

Retarded Fields Verification

D.1 Introduction

In this appendix, we demonstrate the veracity of the manifestly covariant retarded dipole fields (5.69) derived in Section 5.4.5, by showing that they agree completely with the explicit expressions obtained in 1976 by Cohn and Wiebe [54], based on the general expression for the four-potential found in 1967 by Kolsrud and Leer [124]. This task is non-trivial, as the conventions and notation used by Cohn and Wiebe are significantly different from those used throughout this thesis.

D.2 The Cohn–Wiebe field expressions

In this section we list the expressions given for the retarded fields exactly as given by Cohn and Wiebe [54]. The conversion to our conventions and notation will be performed in later sections.

Cohn and Wiebe start with the standard electric charge (or "non-spin") retarded potential in the Lorentz gauge,

$$A^{\mu}_{\text{n.s.}} = -\frac{eV^{\mu}}{R^{\alpha}V_{\alpha}}, \tag{D.1}$$

and the Kolsrud–Leer [124] retarded potential for a particle carrying electric and magnetic dipole moments (or "spin" potential),

$$A^\mu_{\rm s} = \frac{e}{2mR^\alpha V_\alpha}\,\frac{d}{d\tau}\left(\frac{M^{\mu\nu}R_\nu}{R^\beta V_\beta}\right). \tag{D.2}$$

(Kolsrud and Leer [124] did not include the factor $e/2m$ in their expression; also, Cohn and Wiebe have switched the order of the product of R_ν and $M^{\mu\nu}$, effectively introducing an extra minus sign.) It should be noted that this expression, (D.2), differs considerably from that obtained in 1969 by Cohn [53]; the latter result is *wrong*, as pointed out by Kolsrud [125], and admitted by Cohn and Wiebe in the paper we are currently considering [54].

Cohn and Wiebe explicitly note that they are using a $(+,+,+,-)$ metric, world length $d\tau^2 = -g_{\alpha\beta}\,dX^\alpha dX^\beta$, where X^α denotes a field event (*i.e.*, an arbitrary four-position in space). Z^α is used for the particle event (at the retarded time), $V^\mu = dZ^\mu/d\tau$ (this is misprinted as $dX^\mu/d\tau$ in the paper), and $R^\alpha = X^\alpha - Z^\alpha$. The tensor $M^{\mu\nu}$ is simply listed as "the moment tensor characterising the particle"; we shall return to the question of its normalisation value shortly.

For the remaining expressions, the particle is assumed to have no electric dipole moment (in its rest frame), so that $M^{\alpha\beta}V_\beta = 0$. After some manipulations, they find that

$$F^{\mu\nu} = F^{\mu\nu}_{(-1)} + F^{\mu\nu}_{(-2)} + F^{\mu\nu}_{(-3)},$$

where

$$\begin{aligned} F^{\mu\nu}_{(-1)} \equiv -\frac{e}{m\rho^2}\bigg\{ & \frac{3}{2}a_U^2 M^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{1}{2}a_U \dot{M}^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{a_U}{\rho}\dot{M}^{[\nu\alpha}R_\alpha R^{\mu]} \\ & + \frac{1}{2}\dot{a}_U M^{[\nu\alpha}U_\alpha R^{\mu]} - \frac{1}{2}a^2 M^{[\nu\alpha}U_\alpha R^{\mu]} - \frac{1}{2}a_U M^{[\nu\alpha}a_\alpha R^{\mu]} \\ & + \frac{1}{2\rho}\ddot{M}^{[\nu\alpha}R_\alpha R^{\mu]} + m a^{[\nu}R^{\mu]} + m a_U V^{[\nu}R^{\mu]}\bigg\}, \end{aligned}$$

$$
\begin{aligned}
F^{\mu\nu}_{(-2)} &\equiv -\frac{e}{m\rho^2}\left\{\frac{3}{2}\frac{a_U}{\rho}M^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{1}{2\rho}\dot{M}^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{3}{2}a_U M^{[\nu\alpha}U_\alpha U^{\mu]}\right. \\
&\qquad \left. -\frac{1}{2}M^{[\nu\alpha}U_\alpha a^{\mu]} + a_U M^{\mu\nu} + \frac{1}{\rho}\dot{M}^{[\nu\alpha}R_\alpha U^{\mu]} + \dot{M}^{\mu\nu} + mV^{[\nu}U^{\mu]}\right\}, \\
F^{\mu\nu}_{(-3)} &\equiv -\frac{e}{m\rho^3}\left\{\frac{3}{2}M^{[\nu\alpha}U_\alpha U^{\mu]} + M^{\mu\nu}\right\}, \qquad \text{(D.3)}
\end{aligned}
$$

where they have defined the convenient quantities

$$
\begin{aligned}
a^\mu &\equiv \frac{dV^\mu}{d\tau}, \\
\rho &\equiv -R^\alpha V_\alpha, \\
U^\mu &\equiv \frac{R^\mu}{\rho} - V^\mu, \\
a_U &\equiv a_\sigma U^\sigma, \\
\dot{a}_U &\equiv \dot{a}_\sigma U^\sigma, \qquad \text{(D.4)}
\end{aligned}
$$

where the overdot denotes d_τ of the *components* of the vectors in question (*i.e.*, the *partial* derivative, in our notation), and where the sign convention of $F^{\mu\nu}$ in terms of A^μ is given by the usual (but not universal) definition

$$
F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu.
$$

D.3 Conversion of conventions

We now perform the first stage of the conversion of the Cohn and Wiebe fields (D.3), and the corresponding convenient quantities (D.4), by translating the basic *conventions* used.

Clearly, the electric charge expression (D.1) shows that Cohn and Wiebe use an irrationalised system of units. To convert this to the rationalised (naturalised SI) system of units used in this thesis, we need only divide the potentials and fields by 4π.

More important is the fact that they have used a $(+,+,+,-)$ metric, whereas we use a $(+,-,-,-)$ metric. The fact that they call the temporal component X^4, rather than X^0, is irrelevant: they (in contradistinction to Kolsrud and Leer [124]) have not used "*ict*" at all, and so their metric is equally well considered to be $(-,+,+,+)$. Thus, we only need to worry about converting from signature $+2$ to signature -2. To do this, one need only note that the only time that the metric enters into the derivation is whenever one forms a contraction over two indices:

$$(A\cdot B) \equiv g_{\alpha\beta} A^\alpha B^\beta.$$

Thus, we can convert all of the Cohn and Wiebe expressions by performing the transformation

$$(A\cdot B) \longrightarrow -(A\cdot B) \tag{D.5}$$

for any Lorentz dot-product that appears. (It will be noted that their definition of the proper-time τ with a minus sign, with their metric, means that τ needs no conversion.) It is convenient to first convert the quantities (D.4):

$$\begin{aligned} a^\mu &\longrightarrow a^\mu, \\ \rho &\longrightarrow -\rho, \\ U^\mu &\equiv -\frac{R^\mu}{\rho} - V^\mu, \\ a_U &\longrightarrow -a_U, \\ \dot{a}_U &\longrightarrow -\dot{a}_U, \end{aligned} \tag{D.6}$$

where it will be noted that the change in ρ has necessitated a change in the *definition* of U^μ, rather than a transformation of it. Written in terms of these converted quantities (D.6), and performing the conversion (D.5) where necessary, the fields (D.3) become

$$F^{\mu\nu}_{(-1)} = -\frac{e}{4\pi m\rho^2}\left\{-\frac{3}{2}a_U^2 M^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{1}{2}a_U \dot{M}^{[\nu\alpha}U_\alpha R^{\mu]} - \frac{a_U}{\rho}\dot{M}^{[\nu\alpha}R_\alpha R^{\mu]}\right.$$

$$
\begin{aligned}
&\quad + \frac{1}{2}\dot{a}_U M^{[\nu\alpha}U_\alpha R^{\mu]} - \frac{1}{2}a^2 M^{[\nu\alpha}U_\alpha R^{\mu]} - \frac{1}{2}a_U M^{[\nu\alpha}a_\alpha R^{\mu]} \\
&\quad + \frac{1}{2\rho}\ddot{M}^{[\nu\alpha}R_\alpha R^{\mu]} + ma^{[\nu}R^{\mu]} - ma_U V^{[\nu}R^{\mu]}\bigg\}, \\
F^{\mu\nu}_{(-2)} &= -\frac{e}{4\pi m\rho^2}\bigg\{-\frac{3}{2}\frac{a_U}{\rho}M^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{1}{2\rho}\dot{M}^{[\nu\alpha}U_\alpha R^{\mu]} + \frac{3}{2}a_U M^{[\nu\alpha}U_\alpha U^{\mu]} \\
&\quad + \frac{1}{2}M^{[\nu\alpha}U_\alpha a^{\mu]} - a_U M^{\mu\nu} + \frac{1}{\rho}\dot{M}^{[\nu\alpha}R_\alpha U^{\mu]} + \dot{M}^{\mu\nu} + mV^{[\nu}U^{\mu]}\bigg\}, \\
F^{\mu\nu}_{(-3)} &= +\frac{e}{4\pi m\rho^3}\bigg\{-\frac{3}{2}M^{[\nu\alpha}U_\alpha U^{\mu]} + M^{\mu\nu}\bigg\}. \qquad \text{(D.7)}
\end{aligned}
$$

We also rewrite the potentials (D.1) and (D.2) in terms of the converted quantities:

$$A^\mu_{\text{n.s.}} = +\frac{eV^\mu}{4\pi R^\alpha V_\alpha}, \qquad \text{(D.8)}$$

$$A^\mu_{\text{s}} = -\frac{e}{8\pi m R^\alpha V_\alpha}\,\frac{d}{d\tau}\left(\frac{M^{\mu\nu}R_\nu}{R^\beta V_\beta}\right). \qquad \text{(D.9)}$$

D.4 Conversion of notation

We now turn to the task of converting the notation of Cohn and Wiebe into the corresponding notation used in this thesis.

The easiest conversions involve a simple change in symbol. Clearly,

$$
\begin{aligned}
V^\alpha &\longrightarrow U^\alpha, \\
R^\alpha &\longrightarrow \zeta^\alpha, \\
a^\mu &\longrightarrow \dot{U}^\mu, \\
\dot{a}^\mu &\longrightarrow [\ddot{U}^\mu]. \qquad \text{(D.10)}
\end{aligned}
$$

The quantity ρ is also easily recognised: from (D.4), we have

$$\rho \equiv -(R\cdot V) \longrightarrow -(\zeta\cdot U) \equiv -\varphi^{-1}.$$

(Note that the change in metric has already been absorbed into (D.7).) The Cohn and Wiebe U^α, however, must simply be replaced by its corresponding parts: from (D.6) we have

$$U^\mu \equiv -\frac{R^\mu}{\rho} - V^\mu \longrightarrow \varphi\zeta^\mu - U^\mu.$$

We can now compute a_U and $\dot{a}_U$: from (D.4), we have

$$a_U \equiv (a\cdot U) \longrightarrow \varphi(\dot{U}\cdot\zeta) - (\dot{U}\cdot U) \equiv \varphi(\zeta\cdot\dot{U}) \equiv \varphi\dot{\chi},$$
$$\dot{a}_U \equiv (\dot{a}\cdot U) \longrightarrow \varphi[\ddot{U}\cdot\zeta] - [\ddot{U}\cdot U] \equiv \varphi\ddot{\chi} + \dot{U}^2,$$

where in the last expression we have made use of the identity

$$(U\cdot\dot{U}) = 0.$$

D.5 Verification of the retarded potentials

We now verify that the conversions of the expressions used by Cohn and Wiebe for the four-potential A^μ generated by the particle, equations (D.8) and (D.9), agree with the analysis of Chapter 5. (In Chapter 5, the potentials were actually bypassed in favour of the physically observable field strengths $F_{\alpha\beta}$.) A by-product of this verification will be the identification of the remaining symbols used in [54].

Firstly, Cohn and Wiebe's electric charge potential (D.8) is notationally converted using the translations of Section D.4:

$$A^\mu_{\text{CW}} = \frac{e\varphi U^\mu}{4\pi}. \tag{D.11}$$

Using (5.15), (5.22) and (5.19) and integrating, we obtain the equivalent Lorentz-gauge expression for the retarded potential in the notations of Chapter 5,

$$A^\mu_{\text{JPC}} = \frac{q\varphi U^\mu}{4\pi}. \tag{D.12}$$

Comparing (D.11) and (D.12), we see that they agree in functional form. This comparison also requires

$$e \longrightarrow q.$$

This is, indeed, a common use of the symbol e: as that of a general electric charge (*cf.*, *e.g.*, Jackson [113, eq. (14.6)]). However, we shall shortly find that Cohn and Wiebe have *switched conventions* in their later expressions: the charge of the particle will be found, from the field expressions, to be $-e$, *i.e.*, they are thinking of an electron, with the convention that the *positron* charge is $+e$. In anticipation of their change of convention, we shall at this point claim that the identification

$$e \longrightarrow -q \tag{D.13}$$

is correct for the remainder of the Cohn and Wiebe paper.

Turning now to the Kolsrud–Leer potential (D.9), and using the translations of Section D.4, as well as (D.13), we have

$$A^{\mu}_{\text{KL}} = +\frac{q\varphi}{8\pi m}\, d_{\tau}(\varphi M^{\mu\nu}\zeta_{\nu}). \tag{D.14}$$

The "polarisation tensor" $M^{\mu\nu}$ appearing here is related to what was termed the "dipole current tensor", $J^{\mu\nu}$, defined in equation (5.48). Clearly, for a magnetic dipole moment only (and no electric dipole moment), $M^{\mu\nu}$ will be proportional to the normalised spin tensor $\tilde{\Sigma}^{\alpha\beta}$, which is in turn related to the spin vector Σ^{μ} via

$$\tilde{\Sigma}^{\alpha\beta} \equiv \varepsilon^{\alpha\beta\mu\nu} U_{\mu} \Sigma_{\nu}. \tag{D.15}$$

Let us therefore write

$$M^{\mu\nu} \equiv \kappa \tilde{\Sigma}^{\mu\nu}, \tag{D.16}$$

where we shall shortly determine the exact value of κ. The potential (D.14) can then be written

$$A^{\mu}_{\text{KL}} = \frac{\kappa q\varphi}{8\pi m} \varepsilon^{\mu\nu\alpha\beta} d_{\tau}(\varphi U_{\alpha} \Sigma_{\beta} \zeta_{\nu}) \tag{D.17}$$

for a magnetic dipole.

We can obtain the equivalent expression from the analysis of Chapter 5 by using the expression (5.57) and the Lorentz-gauge relation (5.54):

$$A^{\mu}_{\rm JPC} \equiv \partial_\nu A^{\nu\mu} = \frac{\mu}{4\pi}\varepsilon^{\nu\mu\alpha\beta}\partial_\nu \int d\tau\, U_{[\alpha}\Sigma_{\beta]}\,\vartheta(\zeta^0)\,\delta(\zeta^2).$$

Permuting the indices ν and μ in the alternating function by a sign change, absorbing the commutator on the indices of U and Σ by multiplying by a factor of two, using the relation (5.21), integrating by parts, and again using (5.22), we thus find that, in a Lorentz gauge,

$$A^{\mu}_{\rm JPC} = \frac{\mu\varphi}{4\pi}\varepsilon^{\mu\nu\alpha\beta}d_\tau(\varphi U_\alpha \Sigma_\beta \zeta_\nu). \tag{D.18}$$

Comparing (D.18) with (D.17), we see that the functional form is again verified, and furthermore that

$$\frac{\kappa q}{8\pi m} = \frac{\mu}{4\pi},$$

or, in other words,

$$\kappa = \frac{2m\mu}{q}. \tag{D.19}$$

The reason for Kolsrud and Leer for making such a choice can be understood when it is recalled that, for a particle possessing both charge *and* magnetic dipole moment, the magnetic moment is often reëxpressed in terms of the Landé g-factor:

$$\mu \equiv \frac{gq}{2m}s = \frac{gq}{2m}\cdot\frac{1}{2}\hbar \equiv \frac{gq}{4m}$$

(for spin-half particles), and so

$$\kappa = \frac{2m}{q}\cdot\frac{gq}{4m} = \frac{g}{2}.$$

Thus, $M^{\mu\nu}$ is normalised in the same sense as $\tilde{\Sigma}^{\mu\nu}$ for a particle for which $g \equiv 2$ exactly. However, as already noted, the Landé g-factor was invented before neutral particles were discovered, and its use for such particles is messy and contrived; thus, we shall use the results of (D.15), (D.16) and (D.19), namely,

$$M^{\mu\nu} \equiv \frac{2m\mu}{q}\tilde{\Sigma}^{\mu\nu} \equiv \frac{2m\mu}{q}\,\varepsilon^{\mu\nu\alpha\beta}U_\alpha \Sigma_\beta, \tag{D.20}$$

to replace Kolsrud and Leer's $M^{\alpha\beta}$ wherever it appears in favour of μ, U^α and Σ^α.

D.6 Verification of the retarded fields

We can now complete the translation of the Cohn–Wiebe field expressions, (D.7), into the conventions and notation of this thesis.

Concentrating first on the fields generated by the electric charge, we have, from (D.7) and the relations of Section D.4,

$$F' = q\varphi^2(1-\dot{\chi})\zeta\wedge U + q\varphi\zeta\wedge\dot{U}.$$

This agrees with the result (5.28) of Chapter 5, and justifies the identification (D.13).

Turning now to the dipole fields, we need to convert the *magnetic* dipole expressions of Cohn and Wiebe into the equivalent expressions for an *electric* dipole, since we primarily focussed on the electric case in Chapter 5. To do so, we need only note that the electric field per electric dipole unit $\boldsymbol{E}/d$ generated by an electric dipole d^α is equal to the magnetic field per magnetic dipole unit $\boldsymbol{B}/\mu$ generated by a magnetic dipole μ^α (excluding the extra Maxwell term for the magnetic dipole—but Cohn and Wiebe do not consider the worldline fields). Since a duality transformation $F_{\alpha\beta} \to \widetilde{F}_{\alpha\beta}$ induces the transformations $\boldsymbol{E} \to \boldsymbol{B}$, $\boldsymbol{B} \to -\boldsymbol{E}$, we need to take the *negative* of the dual of the magnetic dipole part of the Cohn–Wiebe fields (D.7) in order to get the equivalent electric dipole fields. In other words, we need to compute

$$F^d_n \equiv -\frac{1}{2}\overset{\times}{F}_{(-n)}, \tag{D.21}$$

where $F_{(-n)}$ are the expressions (D.7), by using the relations of Section D.4 to convert the remaining notation. Simplifying the algebra generated by the product of the two alternating functions from (D.21) and (D.20) requires the use of identities (B.44). When this is done, and the various terms collected together, one obtains

$$
\begin{aligned}
F_3^d &= \varphi^3 U\wedge\Sigma - 3\varphi^5\psi\zeta\wedge U, &\text{(D.22)}\\
F_2^d &= \varphi^2\dot{U}\wedge\Sigma + \varphi^3[\zeta\wedge\dot{\Sigma}] + \varphi^3\psi U\wedge\dot{U} - \varphi^3\chi U\wedge\Sigma\\
&\quad + 6\varphi^5\dot{\chi}\psi\zeta\wedge U - 3\varphi^4\dot{\psi}\zeta\wedge U - 3\varphi^4\psi\zeta\wedge\dot{U} + \varphi^3\dot{\vartheta}\zeta\wedge U, &\text{(D.23)}\\
F_1^d &= \varphi^2[\zeta\wedge\ddot{\Sigma}] - \varphi^3\psi[\zeta\wedge\ddot{U}] + \varphi^4\psi\ddot{\chi}\zeta\wedge U - 2\varphi^3\dot{\psi}\zeta\wedge\dot{U} + 3\varphi^4\dot{\chi}\dot{\psi}\zeta\wedge U\\
&\quad - \varphi^3\dot{\chi}[\zeta\wedge\dot{\Sigma}] - \varphi^3\ddot{\eta}\zeta\wedge U - 3\varphi^5\dot{\chi}^2\psi\zeta\wedge U + 3\varphi^4\psi\dot{\chi}\zeta\wedge\dot{U}. &\text{(D.24)}
\end{aligned}
$$

It may be verified that equations (D.22), (D.23) and (D.24) agree precisely with their counterparts (5.70), (5.71) and (5.72) of Section 5.4.5. Thus, the analysis of Chapter 5 agrees completely with that of Cohn and Wiebe [54]; the two quite separate paths taken to obtain the end results mean that one can have confidence in their veracity.

Appendix E

The Interaction Lagrangian

E.1 Introduction

This thesis is concerned with the motion of a single particle, in the classical limit, under the influence of a classical electromagnetic field. In Nature, the only particles for which we expect this to be a very good approximation are the massive leptons: the bare electron, muon and tauon. We also expect that the stable fermionic hadrons, such as the proton and neutron, will also be conveniently described by a single-particle description, as long as the electromagnetic fields involved are sufficiently well-behaved so as not to excite the internal states of these particles. Likewise, the anomalous contributions to the magnetic moments of the massive leptons should be describable quite well by an equivalent Pauli interaction in the single-particle equation, while again keeping in mind that the QED and other processes leading to these contributions are in reality multi-particle effects.

Thus, in the real world, we will generally apply the results of this thesis to spin-half particles. It is therefore of interest to note that, in full generality, there are only *four* possible electromagnetic properties that a spin-half particle may possess: electric charge, electric dipole moment, magnetic dipole moment and anapole moment; each may in general have a structure function. Furthermore, if the spin-half particle does not actually come in contact with any other physical electromagnetic sources, then there are two further simplifications: the anapole moment has no effect whatsoever; and, in the classical limit, the only parts of the structure functions that have any effect are the static values of the moments: q, d and μ.

In the remaining sections of this appendix, we provide brief review of the proofs of these properties, for convenience; there are no new results herein.

E.2 Quantum field theory

Viewed from the perspective of quantum field theory, the electromagnetic interaction of a spin-half particle with an external field is fundamentally a vertex of the fermion with an external photon. Seeing as the photon is fundamentally described by the four-potential, $A(x)$, *any* coupling to it in the Lagrangian must be of the general form $(J \cdot A)$, where J might include operators, such as ∂, when we are viewing the process in position space.

Let the fermion's initial canonical four-momentum (four-wavevector) be b_1, and its final canonical four-momentum b_2. It is convenient to replace these two quantities by the photon's canonical four-momentum, k (which by canonical momentum conservation is equal to $b_2 - b_1$), and the sum of the fermion canonical momenta, $B \equiv b_1 + b_2$. Together with the various matrix quantities characterising the Dirac algebra—namely, 1, γ^5, γ^μ, $\gamma_5\gamma^\mu$, $\sigma^{\mu\nu}$ and $\gamma_5\sigma^{\mu\nu}$—we can proceed to construct the most general coupling to the photon that is possible. Writing out all possible terms blindly, apart from those that are obviously dependent, we have

$$\langle 2|J_\mu(k)|1\rangle = \delta(b_2 - b_1 - k)\,\overline{u}_2\,(F_\mu + G_\mu + H_\mu + K_\mu)\,u_1, \qquad \text{(E.1)}$$

where

$$F_\mu \equiv F_1\gamma_\mu + F_2 k_\mu + F_3 B_\mu + F_4\sigma_{\mu\nu}k^\nu + F_5\sigma_{\mu\nu}B^\nu,$$

$$G_\mu \equiv G_1 \not{k} k_\mu + G_2 \not{k} B_\mu + G_3 \not{B} k_\mu + G_4 \not{B} B_\mu,$$
$$H_\mu \equiv (H_1 \gamma_\mu + H_2 k_\mu + H_3 B_\mu + H_4 \sigma_{\mu\nu} k^\nu + H_5 \sigma_{\mu\nu} B^\nu)\, \gamma_5,$$
$$K_\mu \equiv (K_1 \not{k} k_\mu + K_2 \not{k} B_\mu + K_3 \not{B} k_\mu + K_4 \not{B} B_\mu)\, \gamma_5,$$

and where Lorentz covariance means that the coefficients F_i, G_i, H_i and K_i can only be functions of $k^2 \equiv (k \cdot k)$. We are here using the definitions

$$\{\gamma^\mu, \gamma^\nu\} \equiv g^{\mu\nu},$$
$$\sigma^{\mu\nu} \equiv \frac{i}{2} [\gamma^\mu, \gamma^\nu],$$
$$\gamma_5 \equiv \gamma^5 \equiv \frac{i}{4!} \varepsilon_{\lambda\mu\nu\pi} \gamma^\lambda \gamma^\mu \gamma^\nu \gamma^\pi,$$
$$\not{x} \equiv \gamma^\mu x_\mu,$$

where $[x, y]$ denotes the commutator of x and y and $\{x, y\}$ their anticommutator, and in this section we are using units in which $\hbar = 1$. With the particle on mass shell in the initial and final states, we have

$$(B \cdot k) \equiv b_2^2 - b_1^2 = m^2 - m^2 = 0; \tag{E.2}$$

using also the Gordon identities,

$$\overline{u}_2 \gamma_\mu u_1 = \frac{1}{2m} \overline{u}_2 (B_\mu + i\sigma_{\mu\nu} k^\nu)\, u_1,$$
$$\overline{u}_2 \gamma_\mu \gamma_5 u_1 = \frac{1}{2m} \overline{u}_2 (k_\mu + i\sigma_{\mu\nu} B^\nu)\, \gamma_5 u_1,$$
$$\overline{u}_2 i\sigma_{\mu\nu} B^\nu u_1 = \overline{u}_2 k_\mu u_1,$$
$$\overline{u}_2 B_\mu \gamma_5 u_1 = \overline{u}_2 i\sigma_{\mu\nu} k^\nu \gamma_5 u_1,$$

we can reduce the number of independent terms in (E.1). In particular, we may remove (say) the terms F_3, F_5, H_3 and H_5 as redundant; all of the terms G_i and K_i lead to terms that are either excluded by Lorentz covariance, are included in other terms, or vanish by (E.2), or by virtue of the fact that

$\sigma^{\mu\nu}$ is antisymmetric. The six remaining terms, F_i and H_i ($i = 1, 2, 4$), are further constrained by the requirement of conservation of the fermion current, $(k\cdot J) = 0$. Since

$$\overline{u}_2 \not{k} u_1 \equiv \overline{u}_2 \not{p}_2 u_1 - \overline{u}_2 \not{p}_1 u_1 = 0$$

by the Dirac equation, as does $(k\cdot\sigma\cdot k)$ by symmetry, this requirement has no effect on the terms F_1, F_4 and H_4. However, since $k^2 \neq 0$ in general, the remaining terms must satisfy

$$\overline{u}_2 \left(F_2 k^2 + H_1 \not{k} \gamma_5 + H_2 k^2 \gamma_5 \right) u_1 = 0. \tag{E.3}$$

Now, we can transform the H_1 term by use of the Dirac equation for the incoming and outgoing fermion states, from which one can verify the identity $\overline{u}_2 \not{k} \gamma_5 u_1 \equiv 2m\, \overline{u}_2 \gamma_5 u_1$. For the requirement (E.3) to hold in general, we therefore require $F_2 = 0$ and $2mH_1 + k^2 H_2 = 0$. Replacing H_1 using this result, defining $a(k^2) \equiv H_2(k^2)/2m$, and replacing the $\overline{u}_2 \gamma_5 u_1$ in the H_2 term by $\overline{u}_2 \not{k} \gamma_5 u_1 / 2m$, the general interaction vertex can then be written

$$\begin{aligned}(J\cdot A) = \overline{u}_2 \Big\{ & F_1(k^2)\gamma^\mu + F_4(k^2)\sigma^{\mu\nu}k_\nu + H_4(k^2)\sigma^{\mu\nu}k_\nu\gamma_5 \\ & + a(k^2)\left(\not{k} k^\mu - k^2 \gamma^\mu \right)\gamma_5 \Big\} u_1 A_\mu. \end{aligned} \tag{E.4}$$

E.3 The classical limit

We now investigate what interaction Lagrangian will be obtained from the vertex (E.4) in the classical limit. As we are, in this limit, treating the fermion as a particle, but the photon as a field, it is appropriate to return to position space and replace the photon canonical momentum, k, by the partial derivative of the external potentials and fields, ∂. At the same time, we must replace the various matrix elements in (E.4) by some sort of classical counterparts. It is found that the following transformations produce the desired results:

$$\overline{u}_2 \gamma^\mu u_1 \longrightarrow U^\mu + \frac{1}{2m} \tilde{\Sigma}^{\mu\nu} \partial_\nu,$$

$$\overline{u}_2\sigma^{\mu\nu}u_1 \longrightarrow \tilde{\Sigma}^{\mu\nu},$$

$$\overline{u}_2\sigma^{\mu\nu}\gamma_5 u_1 \equiv \overline{u}_2\varepsilon^{\mu\nu\alpha\beta}\sigma_{\alpha\beta}u_1 \longrightarrow \varepsilon^{\mu\nu\alpha\beta}\tilde{\Sigma}_{\alpha\beta},$$

$$\overline{u}_2\gamma^{\mu}\gamma_5 u_1 \longrightarrow \Sigma^{\mu},$$

where $\tilde{\Sigma}$ is the classical unit spin tensor, and Σ the corresponding spin vector. (The last term in the first expression takes into account the generation of the "Dirac" magnetic moment in the Foldy–Wouthuysen representation, for a particle that has purely an electric charge in the Dirac representation [88].) Since $k \rightarrow \partial$, the structure functions $F_1(k^2)$, $F_4(k^2)$, $H_4(k^2)$ and $a(k^2)$ can be interpreted as functions of the d'Alembertian operator, $\partial^2 \equiv (\partial\cdot\partial)$. We assume that these structure functions are analytic everywhere, so that they can be expanded as a power series in ∂^2; for example

$$a(\partial^2) \equiv a_0 + a_1\partial^2 + a_2\partial^4 + \ldots.$$

We thus find that the most general interaction Lagrangian, in the classical limit, for spin-half particles is given by

$$\begin{aligned} L_{\text{int}} &= F_1(\partial^2)(U\cdot A) + \frac{1}{2}\left\{\frac{F_1(\partial^2)}{2m} + F_4(\partial^2)\right\}(\cdot\tilde{\Sigma}\cdot F\cdot) \\ &\quad - H_4(\partial^2)(\cdot\tilde{\Sigma}\cdot\widetilde{F}\cdot) - a(\partial^2)(\Sigma\cdot J_{\text{ext}}), \end{aligned} \tag{E.5}$$

where we have used the inhomogeneous Maxwell equation,

$$\partial^2 A - \partial(\partial\cdot A) = J_{\text{ext}}$$

to express the $a(\partial^2)$ term in terms of the "external" current $J_{\text{ext}}(x)$ generating the potential $A(x)$, and where $\widetilde{F}$ is the dual electromagnetic field tensor.

Already, one can see in (E.5) the familiar interactions of the classical limit: electric charge; magnetic dipole moment—with its "pure Dirac" and anomalous terms; and electric dipole moment. The final interaction term,

$a(\partial^2)(\Sigma \cdot J_{\text{ext}})$ is recognised as the parity-violating *anapole moment*, first discussed by Zel'dovich [245, 246] in the 1950s. While it is not clear whether the *effective* anapole moment of the real electron (acquired from radiative processes) has any gauge-invariant meaning [103], this interaction nevertheless remains one of the four fundamentally *possible* couplings that a spin-half particle can have.

E.4 Source-free regions

We now consider the above interaction Lagrangian in situations in which the particle under study *does not collide* with any of the electromagnetic sources generating the "external" fields. Most obviously, the anapole interaction term in (E.5) will have no effect, and can be deleted from our considerations. But we also note that the effects of the structure functions $F_1(\partial^2)$, $F_4(\partial^2)$ and $H_4(\partial^2)$ are greatly simplified. To see this, one need only note that all of the interactions apart from that of the electric charge are gauge-invariant, *i.e.* dependent only on the field strengths F and $\widetilde{F}$; furthermore, we know that the classical *equations of motion* derived from the electric charge interaction are also gauge-invariant. Thus, the d'Alembertian operators that occur in the structure functions will only act on the field strengths F and $\widetilde{F}$ in the equations of motion. Let us examine, therefore, the quantities $\partial^2 F$ and $\partial^2 \widetilde{F}$. The former can be simplified using the homogeneous Maxwell equation:

$$\partial^2 F_{\mu\nu} \equiv \partial^\alpha(\partial_\alpha F_{\mu\nu}) = \partial^\alpha(\partial_\mu F_{\alpha\nu} - \partial_\nu F_{\alpha\mu}) \, .$$

But we now note that both of the terms in this last expression involve the quantity $(\partial \cdot F) \equiv J_{\text{ext}}$, by the inhomogeneous Maxwell equation. Thus, in the classical limit, and in source-free regions, $\partial^2 F$ *vanishes identically.* We can similarly apply this analysis to $\partial^2 \widetilde{F}$, but here we need not even require source-free regions, as magnetic charges are incompatible with the vector potential $A(x)$, and do not appear to exist in our universe. Clearly, all higher powers of ∂^2 acting on the fields will also vanish. We are thus left with the drastic simplification that, in source-free regions, the classical electromagnetic interactions of a particle depend *only* on the structure functions' values at $k^2 = 0$.

With this simplification, the surviving terms in (E.5) can be written

$$L_{\text{int}} = q(U \cdot A) + \frac{1}{2}\mu(\cdot \tilde{\Sigma} \cdot F \cdot) - \frac{1}{2}d(\cdot \tilde{\Sigma} \cdot \widetilde{F} \cdot), \tag{E.6}$$

where (reïnstating $\hbar$ explicitly) $q \equiv F_1(0)$, $\mu \equiv q\hbar/2m + F_4(0)$ and $d \equiv 2H_4(0)$; in this thesis, these are referred to simply as *the* (classical) charge, magnetic dipole moment and electric dipole moment respectively.

Finally, we manipulate (E.6) slightly, into a form that is more immediately amenable to analysis for a classical particle. Employing the spin vector Σ^α dual to $\tilde{\Sigma}^{\alpha\beta}$, *i.e.*,

$$\Sigma \equiv \frac{1}{2} U \times \tilde{\Sigma},$$

one can verify that (E.6) becomes

$$L_{\text{int}} = q(U \cdot A) + (\mu \cdot \widetilde{F} \cdot U) + (d \cdot F \cdot U), \tag{E.7}$$

where we have further defined the four-vectors $\mu^\alpha \equiv \mu\, \Sigma^\alpha$ and $d^\alpha \equiv d\, \Sigma^\alpha$. It is this most general classical interaction Lagrangian, either in the form (E.6) or (E.7), that forms the basis of the equations of motion derived in this thesis.

Appendix F

Published Paper

F.1 Introduction

To date, only one paper arising from the work described in this thesis has been submitted and accepted for publication [65]. For completeness, we include here the contents of that paper.

To maintain the integrity of digital copies of this thesis, the paper is included here via the LaTeX source files used to generate the preprint of the paper [64], properly adjusted so that the text herein agrees exactly with the printed form of the paper [65]. For example, the Americanisation of the spelling and punctuation, undertaken by the editors of the journal, is included here verbatim. Corrections made by the author in the proof stage are also included.

For the actual printed form of the paper, please see Volume 9 of the *International Journal of Modern Physics A*. The author will also send paper reprints on request.

International Journal of Modern Physics A, Vol. 9, No. 3 (1994) 461–473

ELECTROMAGNETIC DEFLECTION OF SPINNING PARTICLES

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Received 20 September 1993

We show that it is possible to obtain self-consistent and physically acceptable relativistic classical equations of motion for a pointlike spin-half particle possessing an electric charge and a magnetic dipole moment, directly from a manifestly covariant Lagrangian, if the classical degrees of freedom are appropriately chosen. It is shown that the equations obtained encompass the well-tested Lorentz force and Thomas–Bargmann–Michel–Telegdi spin equations, as well as providing a definite specification of the classical *magnetic dipole force*, whose exact form has been the subject of recent debate. Radiation reaction — the force and torque on an accelerated particle due to its self-interaction — is neglected at this stage.

1. Introduction

The "classical limit" of mechanics has always played an important role in practical physics. While it may be regarded as preferable to describe the behavior of a physical system by solving the problem exactly within quantum mechanics, in practice one need not always do so. If one is only interested in *expectation values* of operators, then one can make good use of Ehrenfest's theorem — which holds good under quite general circumstances (see e.g. Ref. 1, Sec. 31) — and instead solve the equations of *classical* Hamiltonian or Lagrangian dynamics. Thus, for example, the Lorentz force law for charged particles, and the Thomas–Bargmann–Michel–Telegdi spin precession equation[2,3] for particles with spin, are used to advantage every day — implicitly or explicitly — in a wide variety of practical situations.

These two examples, however, owe their widespread acceptance largely to an important quality they possess: they can be derived simply from a knowledge of the external electromagnetic multipolar fields of their pointlike sources, *regardless* of the detailed structure of these sources.[4] On the other hand, the *force* on a pointlike particle with spin is *not* uniquely defined by the external properties of its dipolar electromagnetic fields; this fact, well noted by Thomas in 1927,[2] and more recently by Hraskó,[4] is not yet universally appreciated. To obtain the "correct" force on (and, hence, complete equations of motion for) a particle with a magnetic moment, one must therefore make *some* assumptions, explicit or otherwise, about the nature of the object creating the dipolar field; it is this freedom that has contributed to the ongoing controversy on the subject.[5−9]

It would be impossible to review here all of the various assumptions underlying previous attempts at a complete classical description of particles with spin; instead, we shall simply state the premises and results of a theoretical analysis that we have carried out. In Sec. 2, we outline our considerations in choosing a suitable relativistic classical Lagrangian for a particle with spin. We then, in Sec. 3, outline our derivation of the equations of motion from this Lagrangian, and present the results in what we believe are the most transparent forms. As will be seen, the technique we shall use leads to the omission of all aspects of radiation reaction — which is an omission that should be repaired — but nevertheless the results obtained do allow extensive contact with existing knowledge about the classical behavior of particles with spin.

2. An Appropriate Lagrangian

As already mentioned, some nontrivial assumptions must be made for one to obtain the force equation of motion for even a pointlike particle endowed with a magnetic dipole moment. Our approach is to use a Lagrangian approach from the outset, and hence benefit from the abovementioned guarantee that Ehrenfest's theorem provides. This route is, however, still a potential minefield; as recently emphasized by Barut and Unal,[10] the "spinning top" (or "current loop") phase space degrees of freedom are *not* appropriate to quantum spin-half particles. We shall return to this question shortly.

Our first consideration is to outline precisely *what type* of system we wish to address. In this specification we shall be exacting: we shall only be considering the *intrinsic* magnetic dipole moment of a spin-half, pointlike, structureless particle — together with any electric charge interaction it possesses of course — in externally applied (but, in general, time- and space-varying) electromagnetic fields. It is only with such a tight restriction of our focus that we can proceed confidently in the classical realm at all, as will become apparent. For definiteness, we consider the *electron* a good approximation to the type of particle we are considering; however, we shall not claim any predictive control over contributions from the *anomalous* magnetic moment of the real electron, arising as it does from QED and other processes that are, in reality, additional degrees of freedom over those properly described by the single-particle Dirac equation. We shall not enter the debate as to whether a particle might "intrinsically" have a g-factor differing from 2 in the single-particle Dirac equation; if so, then such "anomalous" moments would be included in our considerations. In any case, since the pure Dirac moment of the electron dominates its anomalous moment numerically, our considerations will, at least, provide useful practical results for real electrons to leading order in the fine-structure constant.

Our main task in this section is to explain the particular Lagrangian that we have chosen to represent the magnetic interaction of such a particle. There are two distinct parts to this decision: firstly, the choice of the *value* that the Lagrangian should take; and secondly, a determination of the appropriate classical *degrees of freedom* that are to be used in the Euler–Lagrange equations — and hence, the functional form of the Lagrangian. The former choice is laid out for us: the Dirac equation gives us

$$L_{\text{int}} = -\frac{gq\hbar}{8m}\sigma^{\alpha\beta}F_{\alpha\beta}, \tag{1}$$

where q is the charge of the particle ($= -e$ for an electron) and g is its gyromagnetic ratio ($= 2$ for the simple minimally-coupled Dirac equation), and our units follow SI conventions with the exception that we set $\varepsilon_0 = \mu_0 = c = 1$. However, to entertain the possibility of the single-particle Dirac equation fully describing a *neutral* particle with an anomalous moment, we shall replace the g factor by the corresponding magnetic moment $\mu = gq\hbar/4m$, thereby separating the interaction of electric charge from that of magnetic moment, at least formally.

Our second task is infinitely more perilous: choosing an appropriate set of generalized coördinates to represent a spin-half particle. Firstly, we observe that the *position* of a particle is usually assumed to be an appropriate quantity in the classical limit; we shall also make this assumption. Thus, we take the (expectation value of the) four-position $z^\alpha(\tau)$ to be four of the independent degrees of freedom. (One can show that, in a general dynamical framework allowing particles with time-dependent masses, all four of these coördinates are indeed independent.) If we take the proper time for the particle, τ, to be the generalized time of the classical Lagrangian framework, then the generalized velocities corresponding to the z^α are given by

$$U^\alpha \equiv \frac{dz^\alpha}{d\tau} \equiv \dot{z}^\alpha. \tag{2}$$

Having defined the four-velocity thus, we can then simply define the *center of energy frame* for the particle as that instantaneous Lorentz frame in which $U^\alpha = (1, 0, 0, 0)$.

We now turn to the *magnetic moment* of the particle. For a spin-half particle, the magnetic moment is parallel to the spin. In the center of energy frame, we expect the particle's spin to be describable by a two-component spinor, of fixed magnitude $s = \frac{1}{2}\hbar$. The expectation value of this spinor is equivalent to a *fixed-length three-vector*, in this frame, whose direction describes simultaneously the expectation value of polarization along any arbitrary axis, as well as the phase difference between the "up" and "down" components along such an axis, but which throws away the *overall* phase of the wavefunction (as does any reduction to the classical limit). It is this *fixed-length* three vector that we shall assume to represent the magnetic moment of the particle.

Consider now the *dynamical* nature of spin angular momentum in the classical limit. It would seem, from the previous paragraph, that the spin should *also* be considered as a fixed-length three-vector in the center of energy frame of the particle. However, this would preclude writing down the usual classical kinetic Lagrangian of rotational motion in terms of generalized *velocities* (i.e. the time derivatives of the Euler angles), namely, in the nonrelativistic limit,[11]

$$L = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}, \tag{3}$$

and thereby preclude obtaining the *torque* on the particle by means of the Euler–Lagrange equations for the rotational degrees of freedom. Our approach, therefore, shall be the following: we shall consider the *spin* of the particle to be given, in the nonrelativistic limit, by its usual form

$$\boldsymbol{s} = \boldsymbol{I} \cdot \boldsymbol{\omega},$$

with the "kinetic" term (3) retained in the Lagrangian in this same limit. This allows the *magnitude* of the spin, $|\boldsymbol{s}| = s$ to be an *a priori* dynamical variable. [We know of course that s should, ultimately, be a constant of the motion (i.e. $\frac{1}{2}\hbar$) if we are to accept the equations of motion as applicable to spin-half particles; we shall discuss this necessity in greater detail shortly.] We shall, on the other hand, still consider the *magnetic moment* of the particle to be formally a function of the generalized coördinates themselves (i.e. the Euler angles) and *not* the velocities. This distinction between the spin and magnetic moment of an electron is an unfamiliar concept within quantum mechanics; indeed, in that situation, no distinction need be made. However, this distinction is vital in the *classical* framework — at least, if one wants to derive the equations of motion from a Lagrangian — as the spin angular momentum and magnetic dipole moment of an arbitrary classical object are not, in general, related in any special way. Our ansatz that the magnetic moment is dependent on the Euler angles alone cannot be justified *a priori*; the only justification will be that it produces equations of motion that satisify the strict requirements of consistency for spin-half particles — that have not, to our knowledge, been satisfied by previous classical Lagrangian approaches. We further note, at this stage, that the ultimate parallelism of spin and magnetic moment *does not* bar us from considering them to have different functional dependencies: the fact that they are parallel may be introduced, in this classical context, as simply a "constitutive" relation, which can only be fully justified upon investigation of the quantum mechanical analysis.

We now turn to synthesizing the information that we have laid out above in a self-consistent, relativistic, Lagrangian framework. Firstly, it is necessary to generalize the nonrelativistic concept of the spin angular momentum $\boldsymbol{s}$ to the relativistic domain. This procedure is carried out in many textbooks on electrodynamics (see e.g. Ref. 12, pp. 556–560); we shall not add anything new. One simply defines a four-vector S^α such that, *in the rest frame* of the particle, it has vanishing zero-component S^0, and its three-vector part $\boldsymbol{S}$ is equal to the nonrelativistic spin $\boldsymbol{s}$. Since, by definition, the three-vector part of the four-velocity, $\boldsymbol{U}$, vanishes in this frame, the identity

$$S^\alpha U_\alpha = 0 \tag{4}$$

shows that only three of the S^α are independent, as one would expect from the nonrelativistic case.

In a completely analogous way, one can generalize the nonrelativistic angular velocity vector $\boldsymbol{\omega}$ and the moment of inertia tensor $\boldsymbol{I}$ to their relativistic counterparts ω^α and $I^{\alpha\beta}$. The nonrelativistic rotational "kinetic" Lagrangian (3) can then be written relativistically as

$$L_{\text{rot}} = \frac{1}{2}\omega^\alpha I_{\alpha\beta}\,\omega^\beta. \tag{5}$$

One might wonder, at this point, how a spin-half particle can have a "kinetic" term of rotation, when it is well known that there is *no* classical "rotating model" that represents spin angular momentum. The answer that the *functional form* of the spin kinetic Lagrangian must be retained in the classical limit, regardless of whether or not it corresponds to any particular "model" that one might dream up. In fact, it will be found that the quantities ω^α and $I_{\alpha\beta}$ will *completely disappear* from the final equations of motion; the only remaining quantity will be the *physically observable* quantity S^α. Thus, by what might arguably be considered a sleight of hand, we can analyze the system in question within Lagrangian mechanics, without recourse to the particular Newtonian models of Refs. 4, 6–8, nor the original heuristic (albeit brilliant) arguments of Thomas[2] and Bargmann, Michel and Telegdi.[3]

Before we can write down the final expression for the Lagrangian we shall use, we must first "massage" the magnetic interaction Lagrangian (1) into a more suitable form. Our major task is to the interpret the spin *tensor*, $\sigma^{\alpha\beta}$, that the Dirac equation introduces. This is not a trivial task: translating between the spin *vector* that we have already defined, S^α, and a spin *tensor*, requires the use of both the alternating tensor $\varepsilon^{\alpha\beta\mu\nu}$, and another four-vector. Often, in quantum mechanics, one uses the *canonical momentum* vector, p^α, for this purpose; the resultant "spin" vector is known as the *Pauli–Lubanski vector*,

$$W^\alpha \equiv \varepsilon^{\alpha\beta\mu\nu} p_\beta S_{\mu\nu}. \tag{6}$$

This quantity is, indeed, very useful in many situations. However, we are here considering *Lagrangian* mechanics; therefore, we should expect that *mechanical* momenta (or in other words, generalized velocities) should be employed; *canonical* momenta belong to Hamiltonian dynamics. We therefore follow Jackson (Ref. 12, p. 556) in using the *four-velocity* to define the transformation between the spin tensor and the spin vector, namely

$$S^\alpha = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu} U_\beta S_{\mu\nu}. \tag{7}$$

It is straightforward to verify that the reverse transformation of (7) is given by

$$S^{\alpha\beta} = \varepsilon^{\alpha\beta\mu\nu} U_\mu S_\nu. \tag{8}$$

Inserting this into (1) and simplifying, we finally obtain our desired magnetic interaction Lagrangian,

$$L_{\text{int}} = \mu^\alpha \widetilde{F}_{\alpha\beta} U^\beta, \tag{9}$$

where $\varepsilon^{0123} \equiv +1$, $\widetilde{F}_{\alpha\beta} \equiv \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}$ is the dual electromagnetic field strength tensor, and the magnetic moment four-vector μ^α is, as noted previously, considered to be a four-vector "embedded" in the intrinsic rotational coördinates of the particle, and hence is functionally dependent on the Euler angles, but *not* their derivatives.

Our complete Lagrangian is then assembled simply from the kinetic rotational term (5), the magnetic interaction term (9), and the standard translational kinetic and electric charge interaction terms:

$$L = \frac{1}{2} m U^\alpha U_\alpha + \frac{1}{2} \omega^\alpha I_{\alpha\beta} \omega^\beta + q U^\alpha A_\alpha + \mu^\alpha \widetilde{F}_{\alpha\beta} U^\beta. \tag{10}$$

It is this Lagrangian, and most particularly its functional form, that forms the basis of the following section.

3. Derivation

We now turn to the question of deriving, from the Euler–Lagrange equations, the equations of motion for the particle under study. (In the following, our language shall describe classical quantities in the same way that Newton would have done, but in reality we are referring to the *expectation values* of the corresponding quantum mechanical operators for the single particle in question.) The generalized coördinates for the particle, following the discussion of the previous section, are taken to be the four translational degrees of freedom z^α, together with the three Euler angles describing the intrinsic "orientation" of the particle.

The mathematical manipulations necessary to obtain the seven Euler–Lagrange equations are in principle no different to those in everyday classical mechanical problems.[11] There is one subtlety, however, that enters into one's consideration of the spin vector S^α and the magnetic moment vector μ^α. By their very definition, these vectors are "tied" to the particle's four-velocity, in the sense that identities such as (4) always holds true. One must therefore be careful when defining their proper-time derivative: since they are, in effect, defined with respect to an *accelerated* frame of reference; there is a difference between taking the time-derivatives of the components of the vector, and the components of the time-derivatives of the vector, as General Relativity teaches us. In fact, the philosophical framework of General Relativity tells us that it is the *latter* that is the generally invariant, "covariant" derivative which should be used in the relativistic Euler–Lagrange equations; the former is simply the "partial" derivative. However, it is straightforward to verify that they can be related via

$$\left(\frac{d}{d\tau} C \right)^\alpha \equiv \dot{C}^\alpha + U^\alpha \left(\dot{U}_\beta C^\beta \right), \tag{11}$$

where C^α is *any* spacelike vector that is orthogonal to the particle's four-velocity (such as S^α and μ^α), the left hand side denotes the "covariant" derivative, and $\dot{C}^\alpha$ the "partial" derivative, of such a vector. (The *Thomas precession*[13,2] is in fact just another way of expressing this "General Relativity" effect — namely, the non-vanishing commutator of Lorentz boosts.)

It is now relatively straightforward to apply the Euler–Lagrange equations to the Lagrangian (10). The three Euler angle degrees of freedom lead, just as in the nonrelativistic case,[11] to the *torque* equation of motion for the particle. With some algebra and simplification, they [together with the identity (4) and the relation (11)] lead to the four-vector equation of motion

$$\dot{S}_\alpha + U_\alpha \dot{U}^\beta S_\beta = F_{\alpha\beta}\mu^\beta + U_\alpha \mu^\nu F_{\nu\beta} U^\beta. \tag{12}$$

It should come as no surprise that this equation is precisely that obtained by Bargmann, Michel and Telegdi [Eq. (6) of Ref. 3], *before* they simply substituted in the Lorentz force law for $\dot{U}_\beta$ [i.e. Eq. (7) of Ref. 3]. As was noted in Sec. 1, this equation — Eq. (12) — is true in general for *any* pointlike object generating an external magnetic dipole field.[4]

We now turn our attention to the four *translational* degrees of freedom of the particle. The Euler–Lagrange equations for these coördinates, for the *same* Lagrangian (10) as used above to generate the Bargmann–Michel–Telegdi equation, immediately yield the four-equation of motion

$$\frac{d}{d\tau}(mU_\alpha) = qF_{\alpha\beta}U^\beta + U^\beta \mu^\nu \partial_\nu \widetilde{F}_{\alpha\beta} + \widetilde{F}_{\alpha\beta}\left(\dot{\mu}^\beta + U^\beta \dot{U}_\nu \mu^\nu\right) + \varepsilon_{\alpha\beta\mu\nu}\mu^\beta U^\mu J^\nu_{\text{ext}}. \tag{13}$$

The first term on the right side is, of course, simply the Lorentz force. The second and third terms, on the other hand, are infinitely more interesting. The second term represents the *gradient* forces on the magnetic dipole moment, i.e. in the nonrelativistic limit, $(\boldsymbol{\mu}\cdot\nabla)\,\boldsymbol{B}$. The third term represents a force of the type $-\dot{\boldsymbol{\mu}}\times\boldsymbol{E}$ in the nonrelativistic limit. The reason that they are so interesting is that many authors have argued that *precisely these terms* should constitute the nonrelativistic limit of the magnetic dipole force (see e.g. Ref. 5 and references therein; also Refs. 4, 7, 8). Here, we have obtained these terms from a relativistic Lagrangian directly, without need for assumptions other than those outlined in the previous section.

The last term in (13) is, as a matter of principle, much deeper, but in practice, completely ignorable. It constitutes a *contact force* between the particle and the electromagnetic current that is generating the "external" field. In practice, such an interaction is usually ignored; however, upon further investigation, it is recognized that it is precisely this contact force that allows the expression (13) to otherwise resemble so closely the force on a "monopole-constructed" dipole model,[4,6] while on the other hand possessing an interaction energy equivalent to a "current loop" model[4,6–8] as required for agreement with atomic hyperfine levels.[14] It is in this way that the "Lagrangian-based" model of this paper seems to select those desirable properties of *both* the "monopole-constructed" and "current-loop" models, while being classically equivalent to neither.

Now that we have dealt with the *dynamics* of the Euler–Lagrange formalism, and have obtained allegedly appropriate equations of motion, we can now specify the exact "constitutive relation" between the magnetic moment and the spin of the particle, for the particular case we wish to study: a spin-half particle. From quantum mechanics, we know that the these two quantities are always parallel, namely,

$$\mu^\alpha = \zeta S^\alpha, \tag{14}$$

where $\zeta \equiv \mu/s$ is some constant, a property of the particle in question, which, for particle of charge q, is commonly written in terms of the g factor as $\zeta \equiv gq/2m$. Before we can use (14), however, we first note that Eq. (13) is itself in a somewhat awkward form: the right hand side involves both $\dot{\mu}^\alpha$ and $\dot{U}^\alpha$. However, the parallelism condition (14) allows us to *uncouple* Eqs. (12) and (13) simply by *substituting* (12) into (13), yielding

$$\frac{d}{d\tau}(mU_\alpha) = qF_{\alpha\beta}U^\beta + U^\beta \mu^\nu \partial_\nu \widetilde{F}_{\alpha\beta} + \zeta\left\{\widetilde{F}_{\alpha\beta}F^\beta{}_\nu \mu^\nu + \widetilde{F}_{\alpha\beta}U^\beta\left(\mu^\sigma F_{\sigma\tau}U^\tau\right)\right\}, \tag{15}$$

where we have, for practical simplicity, dropped the contact term in (13).

There are now several important things we can do with Eqs. (12) and (15). Most importantly, we examine how the *mass* of the particle changes with time; if it is not constant, then our equations of motion cannot possibly apply to any particle, such as an electron or muon, that has a constant rest mass. (It should be noted that the classical relativistic formalism we have employed has made no assumptions as to the constancy of the rest mass m with time: in general, the mass may change as the system in question gains energy from or loses energy to the external field.) It is straightforward to verify that the general proper-time rate of change of the mass of a particle, $\dot{m}$, can be expressed as

$$\dot{m} \equiv U^\alpha \frac{d}{d\tau}(mU_\alpha). \tag{16}$$

Using the the identity (4), and the fact that $\widetilde{F}_{\alpha\beta}F^\beta{}_\nu$ is proportional to the metric, $g_{\alpha\nu}$, it follows that (15) yields, in (16), $\dot{m} = 0$. In other words, the set of Eqs. (12) and (15) *rigorously maintain* the constancy of the mass of the particle. This property is far from trivial; for example, in a classic textbook (Ref. 15, p. 74), a time-varying "effective mass" of the electron was introduced, due to the fact that the equations of motion derived therein allowed *time-changing* rest masses. It is of utmost importance that this difficulty — present in most previous attempts at consistent equations of motion for a dipole — is overcome in the Lagrangian treatment of this paper.

We now verify that the the parallelism condition, (14), is consistent with the equations of motion (12) and (15). The reason for this concern is that we have assumed that the magnetic moment is a constant-magnitude vector, whereas the spin itself is a dynamical quantity that may, in general, *change* its magnitude. We must verify that, if (14) is assumed to hold true at one particular proper time τ, the equation of motion (12) does not change the magnitude of the spin. On diffentiation of the identity $s^2 \equiv -S^\alpha S_\alpha$, one finds $\dot{s} = -\frac{1}{s}\dot{S}^\alpha S_\alpha$; use of (14) in (12) then shows that (12) does, in fact, yield $\dot{s} = 0$. Thus, the magnitude of the spin is a rigorous constant of the motion, just as is the mass.

Our work is now essentially complete. However, to make *practical* use of the equations (12) and (15), it is appropriate to both present them in a more computationally-friendly form, and to highlight clearly where they add to existing knowledge of spin-half particles. These two tasks can essentially be carried out simultaneously. We shall transform (12) and (15) into the standard "$3+1$" form, as, for example, presented in Jackson's textbook (Ref. 12, pp. 556–560). This involves using the three-velocity of the particle in some *particular* fixed "lab" frame, $\boldsymbol{v}$ (Jackson uses the symbol $\boldsymbol{\beta}$), as well as the three-spin $\boldsymbol{s}$ of the particle as seen in its rest frame, but referred to the (nonrotating) coördinates of the "lab" frame. The procedure used to effect this transformation is described in detail in Ref. 12; the algebra is tedious, but straightforward. The results for Eqs. (15) and (12) are

$$\frac{d\boldsymbol{v}}{dt} = \frac{q}{\gamma m}\boldsymbol{E}'' + \frac{g_{\text{eff}}}{\gamma m}\boldsymbol{B}'' + \frac{\Theta}{\gamma^2 m}\boldsymbol{s}' \tag{17}$$

and

$$\frac{d\boldsymbol{s}}{dt} = \boldsymbol{s}\times\boldsymbol{\Omega}_{\text{new}}, \tag{18}$$

where

$$\begin{aligned}\boldsymbol{\Omega}_{\text{new}} = {} & \left\{\zeta - \frac{\gamma-1}{\gamma}\frac{q}{m}\right\}\boldsymbol{B} - \left\{\zeta - \frac{\gamma}{\gamma+1}\frac{q}{m}\right\}\boldsymbol{v}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\left\{\zeta - \frac{q}{m}\right\}(\boldsymbol{v}\cdot\boldsymbol{B})\,\boldsymbol{v} \\ & + \frac{\gamma-1}{\gamma}\frac{g_{\text{eff}}}{m}\boldsymbol{E} + \frac{\gamma}{\gamma+1}\frac{g_{\text{eff}}}{m}\boldsymbol{v}\times\boldsymbol{B} - \frac{\gamma}{\gamma+1}\frac{g_{\text{eff}}}{m}(\boldsymbol{v}\cdot\boldsymbol{E})\,\boldsymbol{v} + \frac{\Theta}{m(\gamma+1)}\boldsymbol{v}\times\boldsymbol{s}\end{aligned}$$

and we have defined the convenient quantities

$$\zeta \equiv \frac{\mu}{s},$$

$$
\begin{aligned}
\Theta &\equiv \zeta^2 \left(\boldsymbol{E}\cdot\boldsymbol{B}\right), \\
\partial' &\equiv \frac{\partial}{\partial t} + \frac{\gamma}{\gamma+1}\left(\boldsymbol{v}\cdot\nabla\right), \\
\boldsymbol{s}' &\equiv \boldsymbol{s} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{v}\cdot\boldsymbol{s}\right)\boldsymbol{v}, \\
\boldsymbol{E}' &\equiv \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{v}\cdot\boldsymbol{E}\right)\boldsymbol{v}, \\
\boldsymbol{E}'' &\equiv \boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \left(\boldsymbol{v}\cdot\boldsymbol{E}\right)\boldsymbol{v}, \\
\boldsymbol{B}'' &\equiv \boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E} - \left(\boldsymbol{v}\cdot\boldsymbol{B}\right)\boldsymbol{v}, \\
\text{and}\quad g_{\text{eff}} &\equiv \zeta\left(\boldsymbol{s}\cdot\nabla\right) + \gamma\zeta\left(\boldsymbol{s}\cdot\boldsymbol{v}\right)\partial' - \gamma\zeta^2\left(\boldsymbol{s}\cdot\boldsymbol{E}'\right),
\end{aligned}
$$

and, in all expressions, the partial derivatives act only on the external field quantities $\boldsymbol{E}$ and $\boldsymbol{B}$.

For ease of comparison with the equations of motion in current usage, we present the Lorentz force law in the same form as (17),

$$
\frac{d\boldsymbol{v}}{dt} = \frac{q}{\gamma m}\boldsymbol{E}'', \tag{19}
$$

and, likewise, the precession frequency vector for the Thomas spin equation (Ref. 12, p. 559):

$$
\boldsymbol{\Omega}_{\text{old}} = \left\{\zeta - \frac{\gamma-1}{\gamma}\frac{q}{m}\right\}\boldsymbol{B} - \left\{\zeta - \frac{\gamma}{\gamma+1}\frac{q}{m}\right\}\boldsymbol{v}\times\boldsymbol{E} - \frac{\gamma}{\gamma+1}\left\{\zeta - \frac{q}{m}\right\}\left(\boldsymbol{v}\cdot\boldsymbol{B}\right)\boldsymbol{v}.
$$

It can be seen that, as advertised, the Lorentz and Thomas equations are contained completely in the new equations. However, several new features are present in both the new force equation, (17), and the new precession frequency vector, $\boldsymbol{\Omega}_{\text{new}}$. Most obviously, the magnetic dipole force is now included in (17), albeit somewhat obscured by the multitude of "convenient quantities" introduced for typographical sanity. A recognition of this expression may again be obtained by taking the nonrelativistic limit (first order in $\boldsymbol{v}$, ignoring Thomas precession and other relativistic effects); (17) then returns us to

$$
\frac{d}{dt}(m\boldsymbol{v}) = q\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right) + \left(\boldsymbol{\mu}\cdot\nabla\right)\left(\boldsymbol{B} - \boldsymbol{v}\times\boldsymbol{E}\right) - \dot{\boldsymbol{\mu}}\times\boldsymbol{E},
$$

which is, as noted earlier, the now generally-accepted[5,7–9] dipole force expression. There are of course numerous new subtleties of (17) that arise from relativistic kinematics; we shall however defer a more exhaustive investigation of them to another place.

Turning now to the spin precession Eq. (18), what may come as a surprise to some is that there are differences between $\boldsymbol{\Omega}_{\text{old}}$ and $\boldsymbol{\Omega}_{\text{new}}$. The reason however is simply found: the Thomas precession effects in the spin precession equation depend on the *acceleration* of the particle; clearly, if we have obtained a more accurate force law, then these changes will necessarily feed through to the spin equation as well. It is interesting to note that the approximate nature of the Thomas expression $\boldsymbol{\Omega}_{\text{old}}$ was noted explicitly by Thomas himself,[2] and by Bargmann, Michel and Telegdi in their rederivation,[3] but that, in the intervening decades, this approximate nature has been lost on many practitioners, who mistakenly believe $\boldsymbol{\Omega}_{\text{old}}$ to be an *exact* expression.

At this point, it is worthwhile commenting on the absence of any *radiation reaction* forces in the equations that have been derived above. Their omission can, in fact, be traced back directly to the (implicit) assumption that the electromagnetic potentials and fields in the Lagrangian (10) are the *externally generated ones only*. In the case of the electric charge interaction, it was recognized by Lorentz,[16] and later clearly explained by Heitler,[17] that this is an incorrect assumption. For full consistency, one must include the potentials and fields *of the particle itself* in the Lagrangian, even though they appear, at first sight, to be hopelessly divergent quantities. If one proceeds with extreme care, one can show that the effects on the equations of motion of this "self-interaction" are, in fact, twofold: firstly, the addition of the (finite) radiation reaction terms of the Lorentz–Dirac equation; and secondly, the *dynamical* effects explaining the electromagnetic self-energy contributions to the rest mass of the system, for whatever arbitrary charge distribution is assumed. The beauty of this procedure is that it reveals the all-encompassing nature of the Lagrangian; no further input is required to obtain the equations of motion.

The success of the above procedure clearly indicates that a similar procedure should be undertaken for the additional magnetic moment interaction Lagrangian present in (10). Barut and Unal[10] have considered the dipole radiation reaction question from the point of view of a semi-classical *Zitterbewegung* model, but to our knowledge an exact treatment of this problem in terms of the classical spin vector $\boldsymbol{s}$ has not been performed. This problem is one that we are currently investigating. However, it is vastly more complicated than the electric charge case, by virtue of the inclusion of *rotatonal* degrees of freedom for the particle. In addition, one already knows in advance that the electric charge and magnetic moment *interact* in their respective radiative terms, as is evidenced by the Sokolov–Ternov effect[18] (the polarization of electrons due to their emitted synchrotron radiation), spectacularly confirmed in the polarization experiments at LEP in recent years.[19,20] Any prospective solution of the combined radiation reaction equations for a charged, spinning particle must therefore, at the very least, reproduce this important result.

4. Conclusions

It has been shown in this paper that it *is* possible to construct a fully consistent, comprehensive, relativistic classical Lagrangian framework for analyzing the motion of spinning particles possessing both electric charge and magnetic dipole moments. The results obtained here are not revolutionary. They encompass the well-known Lorentz force and Thomas–Bargmann–Michel–Telegdi equations. They provide a rigorous foundation for the magnetic dipole force law currently believed to be the most appropriate for such particles. They further integrate, seamlessly, this dipole force with the Lorentz force and Thomas–Bargmann–Michel–Telegdi equations, in a fully relativistic way.

It should be noted that the results of this paper *agree* with the lowest-order terms obtained in the analysis of Anandan, based on the Dirac equation[21] — and of particular note, the "Anandan force" proportional to $\boldsymbol{E} \times (\boldsymbol{\mu} \times \boldsymbol{B})$, which is of course simply the zeroth-order term in the force term $-\dot{\boldsymbol{\mu}} \times \boldsymbol{E}$ above, when it realized that $\dot{\boldsymbol{\mu}}$ is, via the Thomas–BMT equation, proportional to $\boldsymbol{\mu} \times \boldsymbol{B}$ to zeroth order. The current work, however, includes terms to *all* orders in the particle's velocity, not just the lowest-order limit of Anandan. It should also be noted that the criticisms of Casella and Werner[22] of Anandan's analysis[21] are *erroneous*, being based on an obvious omission of all "spin-flip" terms from their quantum mechanical equations of motion.

The framework outlined in this paper may now be used as a platform for full inclusion of radiation reaction — not just for the electric charge, but also for the magnetic moment, and their mutual interactions — in the classical limit. If the enormous assistance provided by the existing classical radiation reaction theory, both in terrestrial and astrophysical applications, is any guide, then one can only speculate what additional richness of physical phenomena will be made sensible with this addition to our analytical resources.

Acknowledgments

Helpful discussions with I. Khriplovich, J. Anandan, N. Mukhopadhyay, L. Wilets, P. Herczeg, A. G. Klein, G. I. Opat, J. W. G. Wignall, A. J. Davies, G. N. Taylor, K. Jones, R. E. Behrend and S. Bass are gratefully acknowledged. This work was supported in part by the Australian Research Council, an Australian Postgraduate

Research Allowance and a Dixson Research Scholarship. We warmly thank the Institute for Nuclear Theory at the University of Washington for its hospitality and the United States Department of Energy Grant #DOE/ER40561 for partial support during the completion of this work.

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Appendix G

Computer Algebra

Mrs. Dunn looked at her son with bewilderment.

"I don't undertand. How can a computer do your homework?"

"Well, first we feed all the information from our schoolbooks into it, then we analyze the problems we have for homework, and we program them, and we let the machine solve them and type them out for us."

—— "Danny Dunn and the Homework Machine" (1958) [241]

G.1 Introduction

The author, reading such words in 1974, found them tantalisingly exciting, but somewhat far-fetched. Twenty years later, the author finds himself fetching himself from afar.

This appendix contains the output listings of various computer algebra programs, written by the author, that were used to both perform the lengthy vector-algebraic calculations required for this thesis, and to put these expressions into a LaTeX-readable format consistent with the macro set used by the author.

Note that the output from several of the programs essentially serve as appendices in their own right, and their typographical and linguistic format are indistinguishable from that generated manually by the author.

G.2 Description of the programs

There are five computer algebra programs, written by the author, that have been used in the preparation of this thesis: RADREACT, KINEMATS, RETFIELD, TEST3INT and CHECKRS.

A brief description of each program, and the reasons for its existence, follow.

G.2.1 RADREACT: Radiation reaction

When the author first made an attack on the radiation reaction calculations of Chapter 6, he expanded the Taylor series (2.84) to *two fewer* orders than now appear. This was sufficient to compute the electric charge radiation reaction self-force—*i.e.*, the Lorentz calculation, corrected by the author with the gravitational redshift factor;—and, while more algebraically complicated than the naïve Lorentz model of Galilean rigidity, the expressions involved were not overly onerous.

The successful correction of the Lorentz derivation led to plans of its extension to the dipole moments. The first stage, of course, was the computation of the explicit retarded fields from the dipole—now presented in Chapter 5; this had not, at the time, yet been performed. Once it was found that they were obtainable in a reasonably simple form, the radiation reaction calculations were begun. It was immediately clear that, due to the $1/R^3$ rather than $1/R^2$ variation of the static fields, an extra order would be required in the calculations. This was begun.

During this task, further comtemplation revealed that the only practical way of computing the *gradient* dipole forces—other than the unpalatable prospect of computing the retarded four-gradients explicitly, and painfully trying anew to simplify *them*—was by including yet *another* order of ε in the retarded field expressions, and then using the explicit $\boldsymbol{r}_d$ and $\boldsymbol{r}_s$ (and hence

$\boldsymbol{r}$) dependence to extract the three-gradient at $t = 0$. That this would begin to blow out the length of the algebraic expressions involved is recognised simply by considering the combinatorics of the available three-vector quantities. Nevertheless, the calculations in progress (to one lower of ε than ultimately required) were completed, for the kinematical expressions listed explicitly in Chapter 6. From there, the retarded fields *of the electric charge only* were expanded to this extra order in ε. This allowed a computation of *all* of the radiation reaction effects due to the electric charge fields, since there was now sufficient orders of ε to compute the gradient dipole forces.

When this was done, the "spin renormalisation" effect, described in Chapter 6, was one consequence. At first, this was perplexing. However, once the origin in terms of the static fields was clearly understood—and, moreover, agreement was found between the expression derived from the radiation reaction calculations and that from first principles—the feeling that the computations being undertaken were *indeed* physically correct was cemented.

The results, to this point, had all been extensively cross-checked. All calculations were carried out twice, at separated intervals. The crucial three-vector Taylor series expressions have a nice consistency check: the quantity $\gamma(\tau)$ may be computed two ways: from $\boldsymbol{v}(\tau)$, or by taking the derivative $d_\tau t(\tau)$; these results agree. Quantities such as $\boldsymbol{n}$, $\boldsymbol{n}'$ and $\boldsymbol{n}''$ were dot-producted together, and the results compared with that expected (for example, $\boldsymbol{n}^2 = 1$). The high degree of cancellation of a number of the terms in the integrated results also indicated that the calculations were most likely sound.

The task of extending these calculations to another order in ε were then begun. They were carried out—and double-checked—up to the calculation of $\boldsymbol{n}$. However, at this point, the combinatorical explosion began to wear down the author. Simply writing out expressions with more than seventy terms, with half a dozen or more factors in each, was excessively time-consuming. Moreover, little insight is required from that point, to the end of the computations: they are simply algebraic manipulations (with some—but not too much—subtlety required to handle the three-vector gradients and integrations), more suited to a computer than a human. (This is to be contrasted with the the calculation of the Taylor series coëfficients, which is almost indecipherable if done the naïve way, but most manageable if iterated order by order.) Thus, it was suggested by the author's supervisor, Professor McKellar, that a computer algebra program such as Mathematica be used to perform these calculations.

Investigations were made into this possibility. Two major obstacles were found. Firstly, it did not appear that Mathematica possessed sufficient inherent intelligence to handle three-vectors in the abstract way required for these calculations. This could, of course, be programmed in, but that would require a substantial programming investment, in a language the author was totally unfamiliar with, at a very late stage of the game. Secondly, it was not clear how the output of Mathematica's manipulations could be compared with the existing hand-computed results, on the one hand; or integrated into the LaTeXable body of this thesis, on the other; without a further substantial programming (or painful transcription) process. Investigation into other available computer algebra systems showed possibility for alleviating the first problem, but the second remained. There was also the logistical problem that, while Mathematica is widely used throughout the University, and hence help and support is readily available, the use of other packages would be a major task, undertaken essentially alone, and from scratch.

For these reasons, the author decided to write a simplified computer algebra system, that would be powerful enough to perform the computations actually required for this thesis, without further bells or whistles; and, equally important, the system would output the results using *exactly the same macros* that the author has used to prepare the remainder of this thesis. Familiarity with the two obvious language contenders, ANSI C and C++, yielded a difficult choice; despite the elegance of C++ for such tasks, ANSI C was ultimately chosen, both for simplicity, and to ensure a standard result that could be run on any machine—the author already having been burnt previously by the implementation-dependencies of C++, which has not, unfortunately, been standardised as yet.

Of course, the output of computer programs must always be eyed suspiciously, being only as reliable as the programmer's concentration level at the most fatigued point of his work. It is for this reason that the fully checked, manually-obtained results for the electric charge field contributions to the radiation reaction terms—computed from beginning to end—were crucial, for the purpose of debugging the program at each stage of its development. The computations up to $\boldsymbol{n}$ of the final-order-in-ε results were also important, in order to verify that the algorithms were correctly handling the extra order of complexity involved. Once the integrity of the system had been exhaustively verified, the remaining computations were performing exclusively by computer—being, of course, too lengthy to continue further on paper.

At a late stage, it was recognised that the definition of the relativistically rigid body originally used by the author did not correctly handle the subtle *Thomas precession* of the constituents, as the body made the transition to a finite velocity and acceleration (*see* Section 3.3). This rendered almost all of the author's manually-computed results erroneous, in the lower orders. Fortunately, the program RADREACT was completed before this oversight was noted, and it was quite a minor task adding the extra precession expression to the program.

When the program was then run anew, it was found that, indeed, practically all expressions which contained sufficient orders in τ or ε (specifically, those for which terms involving both $\dot{\boldsymbol{v}}$ and $\ddot{\boldsymbol{v}}$ were present) were changed by the redefinition of the rigid body. However,—and most surprisingly,—*the final radiation reaction equations of motion were not changed at all*, despite the fact that even the *penultimate* equations had significant modifications. Although the deep physical meaning behind such an "invariance" of the results to changes in the rigid body definition was mysterious, the fact that the computer algebra program *did* indeed return to the same answer—by a somewhat circuitous path—indicated to the author that the program was, at the least, robust.

Of course, the program still relies on the author's assumptions; no amount of robustness would counter the effects of an erroneous formulation. The reader is left to decide for themselves if the author's framework is justifiable.

G.2.2 KINEMATS: Kinematical quantities

Following the successful creation of most of the libraries required for the program RADREACT, the author realised that there would be little extra overhead involved in using those same libraries to verify the other, quite complicated algebraic results obtained prior to the attack on the radiation reaction problem. In particular, there were numerous kinematical identities that had been computed and extensively cross-checked by the author, on paper, that were at the time collected together into an appendix titled "Kinematical Quantities"; and there were the author's greatly simplified explicit expressions for the retarded dipole fields, given (then, as now) in Chapter 5.

Thus, the program KINEMATS was written to verify the kinematical results collected in the appendix "Kinematical Quantities". It was found that all results were (as expected) correct.

However, when the program RADREACT was nearing completion, it was found that extra kinematical results were required, over and above those originally computed. It was at this stage found most time-efficient to compute these using the (by now fully debugged) program KINEMATS, bypassing a manual evaluation altogether.

Faced with the task of grafting these new results into the existing separate appendix, the author realised that it it would be simpler to scrap the original appendix altogether, and instead use the output of KINEMATS as the sole appendical reference. In following this course of action, the output of the program was brought up to the same standards as those of the appendix it was replacing: extra explanatory text was added, equations were numbered, and the results collected together into convenient subsections. The result is that the output from KINEMATS is now very close to the original appendix,—together with extra results that would be intractably complicated if computed by hand,—with the added advantage that the possibility of human error in the transcription of mathematical results has been eradicated.

G.2.3 RETFIELD: Retarded fields

After the program KINEMATS was completed, the author proceeded to write the program RETFIELD, to verify that the simplified retarded dipole field expressions obtained (after much pain) by the author, on paper, were in fact correct.

To do so, the program first extracts the electric and magnetic parts of the electromagnetic field strength tensor $F_{\alpha\beta}$, using the results computed in KINEMATS, in terms of the three-vectors $\boldsymbol{n}$, $\boldsymbol{v}$, $\boldsymbol{\sigma}'$, and their lab-time derivatives; $\boldsymbol{n}'$ and $\boldsymbol{n}''$ are not used. The program then expands out the $\boldsymbol{n}'$ and $\boldsymbol{n}''$ appearing in the author's expressions, also in terms of these quantities. The two sets of results are then compared.

This verification process turned out to be surprisingly straightforward, using the computer algebra libraries already written; the author's expressions were, in fact, verified, without need for any modifications or corrections.

G.2.4 TEST3INT: Testing of 3-d integrations

The program TEST3INT was written to test the various three-dimensional integration routines invoked in the final step of the radiation reaction calculations.

Because TEST3INT was used only for verification of integrity purposes, the output is straight ASCII text, rather than LaTeX source text.

G.2.5 CHECKRS: Checking of inner integrals

The program CHECKRS does not actually use any of the computer algebra libraries used for the above programs, but is instead a rudimentary numerical integration program that computes an result required in the verification of the $\boldsymbol{r}_s$ integrals in Chapter 6. It is listed here along with the other ANSI C programs for convenience.

G.3 Running the programs

Although all of the results of the computer algebra program are listed explicitly in this thesis, some readers may be interested in running the programs themselves, to observe the computations being performed first-hand.

To do so, one requires the following: a computer system with an ANSI C compiler installed; a computer system with a "big" implementation of TeX and LaTeX installed (not necessarily the same system); and the thirty-three files listed in Table G.1 that are included in digital copies of this thesis. One also needs to know how to carry out the instructions below on the computer systems in question.

The twenty-four files ending with .C and .H must be copied to the computer system equipped with the ANSI C compiler.

Table G.1 should be consulted to determine which .C files are required for the program desired; these should be compiled, and the resulting object files linked.

The resultant binary file should then be run; the program will output text both to the display device, and simultaneously to the relevant file listed in the 'Output filename' column of Table G.2. This output file, together with

Program	Filename	Description
KINEMATS	KINEMATS.H	Header file.
	KINEMATS.C	Source code.
RETFIELD	RETFIELD.H	Header file.
	RETFIELD.C	Source code.
RADREACT	RADREACT.H	Header file.
	RADREACn.C	($n = 1$ to 5) Source code.
TEST3INT	TEST3INT.C	Source code.
CHECKRS	CHECKRS.C	Source code.
All	ALGEBRA.H	Header file for computer algebra library.
	ALGEBRAn.C	($n = 1$ to 6) Computer algebra library.
	FRACTION.H	Header file for fraction library.
	FRACTION.C	Fraction library.
	LATEXOUT.H	Header file for LaTeX output library.
	LATEXOUT.C	LaTeX output library.
	MISCUTIL.H	Miscellaneous utilities.
LaTeX	MACROS.TEX	Macro set for thesis.
	AMERICAN.TEX	American spelling and punctuation choices.
	BRITISH.TEX	British spelling and punctuation choices.
	COSTELLA.TEX	The author's spelling and punctuation choices.
	KM.TEX	Skeleton file for program KINEMATS.
	RF.TEX	Skeleton file for program RETFIELD.
	RR.TEX	Skeleton file for program RADREACT.
	T3.TEX	Skeleton file for program TEST3INT.
	CRS.TEX	Skeleton file for program CHECKRS.

Table G.1: The thirty-three files, included in digital copies of this thesis, required to run the computer algebra programs described in this appendix.

Program	Output filename	Skeleton filename
KINEMATS	KMOUTTH.TEX	KM.TEX
RETFIELD	RFOUTTH.TEX	RF.TEX
RADREACT	RROUTTH.TEX	RR.TEX
TEST3INT	T3OUTTH.TXT	T3.TEX
CHECKRS	CRSOUTTH.TXT	CRS.TEX

Table G.2: Output and skeleton filenames for the five ANSI C programs. (See description in text.)

the corresponding file listed in the 'Skeleton filename' column of Table G.2, and the first four files listed in the 'LaTeX' section of Table G.1, should be copied to the computer system installed with TeX and LaTeX. The 'skeleton file'—which simply includes the output file and enables processing as a standalone section—should then be LaTeXed, using the spelling and punctuation conventions desired (AMERICAN, BRITISH or COSTELLA). The resulting .DVI file is the fully-formatted output of the program, which can then be printed or viewed as desired.

If one encounters problems with the above use of the 'skeleton file', the output from the three programs that produce LaTeX source code (KINEMATS, RETFIELD and RADREACT) may be processed directly. To do so, one needs to add the `\StartChap` and `\SingleSection` commands at the start of the file (after the file MACROS.TEX is included), and the `\EndChap` command to the end of the file. The output files from the remaining two programs are plain ASCII text.

The following sections contain the fully-LaTeXed output of the five computer algebra programs KINEMATS, RETFIELD, RADREACT, TEST3INT and CHECKRS.

G.4 KINEMATS: Kinematical quantities

G.4.1 Introduction

This program computes various formulæ for kinematical quantities used in this thesis.

G.4.2 Lorentz transformation

If a four-vector X has explicit components X^α in some lab frame, then a boost of X by the three-velocity $\boldsymbol{v}$ transforms its components into X'^α, according to

$$\begin{aligned} X'^0 &= \gamma X^0 + \gamma(\boldsymbol{v}\cdot\boldsymbol{X}), \\ \boldsymbol{X}' &= \boldsymbol{X} + \gamma X^0 \boldsymbol{v} + \gamma^2(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{X})\boldsymbol{v}, \end{aligned} \tag{G.1}$$

where every definition of a three-velocity $\boldsymbol{v}$ carries with it the implicit definition of a corresponding *gamma factor*,

$$\gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{v}^2}}. \tag{G.2}$$

One may verify that

$$X'^2 = X^{0^2} - \boldsymbol{X}^2.$$

The components X'^α may be interpreted as being the components of X in a "primed lab frame". The primed lab frame sees the original lab frame as moving with three-velocity $+\boldsymbol{v}$, *i.e.*, the primed frame is moving with three-velocity $-\boldsymbol{v}$ with respect to the original frame.

G.4.3 Four-velocity of a particle

The four-velocity, U, of a particle has components

$$U \equiv (1, \boldsymbol{0}) \tag{G.3}$$

in the MCLF of the particle; in a frame in which it moves with three-velocity $\boldsymbol{v}$, application of the Lorentz transformation (G.1) yields

$$\begin{aligned} U^0 &= \gamma, \\ \boldsymbol{U} &= \gamma \boldsymbol{v}. \end{aligned} \tag{G.4}$$

One may verify that

$$U^2 = 1.$$

G.4.4 Four-spin of a particle

The four-spin, Σ, of a particle has components

$$\Sigma \equiv (0, \boldsymbol{\sigma}) \tag{G.5}$$

in the MCLF of the particle; in a frame in which it moves with three-velocity $\boldsymbol{v}$, application of the Lorentz transformation yields

$$\begin{aligned} \Sigma^0 &= \gamma(\boldsymbol{v}\cdot\boldsymbol{\sigma}), \\ \boldsymbol{\Sigma} &= \boldsymbol{\sigma} + \gamma^2(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}. \end{aligned}$$

One may verify that

$$\Sigma^2 = -1.$$

G.4.5 The FitzGerald three-spin

We [65] introduce a further quantity, derived from the three-spin $\boldsymbol{\sigma}$, that considerably simplifies a number of explicit algebraic expressions: the *FitzGerald three-spin*,

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \gamma(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}. \tag{G.6}$$

The reason for us naming it after FitzGerald can be seen by computing its three-magnitude:

$$\boldsymbol{\sigma}'^2 = 1 - (\boldsymbol{v}\cdot\boldsymbol{\sigma})^2.$$

The magnitude of $\boldsymbol{\sigma}'$ is (like that of $\boldsymbol{\sigma}$) unity, if $\boldsymbol{\sigma}$ lies in a plane perpendicular to the three-velocity $\boldsymbol{v}$; but it is *contracted* by a factor of

$$\sqrt{1-\boldsymbol{v}^2} \equiv \frac{1}{\gamma}$$

if $\boldsymbol{\sigma}$ lies parallel or antiparallel to the direction of $\boldsymbol{v}$; in other words, $\boldsymbol{\sigma}'$ acts like a *FitzGerald–Lorentz contracted* [87] version of $\boldsymbol{\sigma}$.

It is curious, but probably not fundamentally meaningful, that a concept from the pre-relativistic days should be a useful tool in a completely relativistic analysis.

G.4.6 Proper-time and lab-time derivatives

The proper-time rate of change experienced by a particle moving with four-velocity U, of an arbitrary quantity external to the particle (*e.g.*, an external field), is computed by means of the relativistic convective derivative operator,

$$d_\tau \equiv (U\cdot\partial) \equiv \gamma\partial_0 + \gamma(\boldsymbol{v}\cdot\nabla). \tag{G.7}$$

The proper-time rate of change of a component of a kinematical property of a particle (*i.e.*, the partial proper-time derivative), as seen in some given lab frame, is defined to be related to the lab-time rate of change of that component by means of the time-dilation formula:

$$[d_\tau] \equiv \gamma d_t. \tag{G.8}$$

G.4.7 Partial kinematical derivatives

We now take successive proper-time derivatives of the *components* of U and Σ (*i.e.*, *partial* proper-time derivatives). To do so, we first take d_t of (G.2) to find

$$\dot{\gamma} \equiv \gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}). \tag{G.9}$$

Using (G.8) and (G.9), and taking note of the overdot conventions, we thus find

$$\begin{aligned}
[\dot{U}^0] &= \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}), \\
[\dot{\boldsymbol{U}}] &= \gamma^2\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}$$

$$\begin{aligned}
[\ddot{U}^0] &= \gamma^5\dot{\boldsymbol{v}}^2 + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}) + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2, \\
[\ddot{\boldsymbol{U}}] &= \gamma^3\ddot{\boldsymbol{v}} + \gamma^5\dot{\boldsymbol{v}}^2\boldsymbol{v} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v},
\end{aligned}$$

$$\begin{aligned}
[\dddot{U}^0] &= \gamma^6(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}}) + 3\gamma^6(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + 28\gamma^{10}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3 + 13\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) \\
&\quad + 13\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}), \\
[\dddot{\boldsymbol{U}}] &= \gamma^4\dddot{\boldsymbol{v}} + 4\gamma^6\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + 6\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + 4\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^6(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^6(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 19\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + 28\gamma^{10}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3\boldsymbol{v} + 13\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 13\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}$$

$$\begin{aligned}
[\ddddot{U}^0] &= 3\gamma^7\ddot{\boldsymbol{v}}^2 + 13\gamma^9\dot{\boldsymbol{v}}^4 + \gamma^7(\boldsymbol{v}\cdot\ddddot{\boldsymbol{v}}) + 4\gamma^7(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + 13\gamma^9(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})^2 \\
&\quad + 280\gamma^{13}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^4 + 26\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}) + 188\gamma^{11}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2 \\
&\quad + 19\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}}) + 57\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + 188\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}), \\
[\ddddot{\boldsymbol{U}}] &= \gamma^5\ddddot{\boldsymbol{v}} + 10\gamma^7\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + 3\gamma^7\ddot{\boldsymbol{v}}^2\boldsymbol{v} + 13\gamma^9\dot{\boldsymbol{v}}^4\boldsymbol{v} + 10\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + 10\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&\quad + 5\gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^7(\boldsymbol{v}\cdot\ddddot{\boldsymbol{v}})\boldsymbol{v} + 15\gamma^7(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 4\gamma^7(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 55\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 13\gamma^9(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{v} + 180\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + 280\gamma^{13}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^4\boldsymbol{v} \\
&\quad + 75\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 26\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 188\gamma^{11}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} \\
&\quad + 75\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 19\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\boldsymbol{v} + 57\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 188\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}$$

$$
\begin{aligned}
[\dot{\Sigma}^0] &= \gamma^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) + \gamma^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}),\\
[\dot{\boldsymbol{\Sigma}}] &= \gamma\dot{\boldsymbol{\sigma}} + \gamma^3(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \gamma^3(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&\quad + \gamma^3(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 2\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad - \gamma^6(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
[\ddot{\Sigma}^0] &= \gamma^3(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}) + 2\gamma^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + \gamma^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + \gamma^5\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}),\\
[\ddot{\boldsymbol{\Sigma}}] &= \gamma^2\ddot{\boldsymbol{\sigma}} + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + 2\gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&\quad + \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v} + 2\gamma^4(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&\quad + 2\gamma^4(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} + \gamma^4(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad + 2\gamma^6(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} - \gamma^7(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad + 5\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 5\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&\quad + 5\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 2\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad - 2\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 2\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&\quad - 2\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} - \gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad + 10\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 8\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&\quad + 2\gamma^{10}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
[\dddot{\Sigma}^0] &= \gamma^4(\boldsymbol{v}\cdot\dddot{\boldsymbol{\sigma}}) + 3\gamma^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) + 3\gamma^4(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + \gamma^4(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 4\gamma^6\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad + 4\gamma^6\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 6\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}) + 12\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad + 6\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 4\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) + 4\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&\quad + \gamma^6(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 3\gamma^6(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + 19\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\\
&\quad + 19\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 28\gamma^{10}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 13\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\\
&\quad + 13\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}),\\
[\dddot{\boldsymbol{\Sigma}}] &= \gamma^3\dddot{\boldsymbol{\sigma}} + \gamma^5\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}\\
&\quad + \gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} + 3\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+3\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&+3\gamma^5(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+6\gamma^5(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+3\gamma^5(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v}+3\gamma^5(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+3\gamma^5(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}+\gamma^5(\gamma+1)^{-1}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+7\gamma^7(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+7\gamma^7(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&+7\gamma^7(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-3\gamma^8(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+9\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+18\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+9\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v}+18\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+18\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}+9\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+7\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+7\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&+7\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}+2\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+6\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-6\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&-6\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-6\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&-\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}-3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v}\\
&+40\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+40\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&+40\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}-27\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-27\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}-27\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&+6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}+80\gamma^{11}(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&-82\gamma^{12}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}+36\gamma^{13}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&-6\gamma^{14}(\gamma+1)^{-4}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}+32\gamma^9(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}
\end{aligned}
$$

$$
\begin{aligned}
&-25\gamma^{10}(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}+6\gamma^{11}(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+32\gamma^{9}(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&-25\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

Evaluating all these quantities for $\boldsymbol{v}=\mathbf{0}$, we find

$$
\begin{aligned}
U^0\Big|_{v=0} &= 1,\\
\boldsymbol{U}|_{v=0} &= \mathbf{0},\\[2ex]
[\dot{U}^0]\Big|_{v=0} &= 0,\\
[\dot{\boldsymbol{U}}]\Big|_{v=0} &= \dot{\boldsymbol{v}},\\[2ex]
[\ddot{U}^0]\Big|_{v=0} &= \dot{\boldsymbol{v}}^2,\\
[\ddot{\boldsymbol{U}}]\Big|_{v=0} &= \ddot{\boldsymbol{v}},\\[2ex]
[\dddot{U}^0]\Big|_{v=0} &= 3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}),\\
[\dddot{\boldsymbol{U}}]\Big|_{v=0} &= \dddot{\boldsymbol{v}}+4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}},\\[2ex]
[\ddddot{U}^0]\Big|_{v=0} &= 13\dot{\boldsymbol{v}}^4+3\ddot{\boldsymbol{v}}^2+4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}),\\
[\ddddot{\boldsymbol{U}}]\Big|_{v=0} &= \ddddot{\boldsymbol{v}}+10\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}+15(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}},\\[2ex]
\Sigma^0\Big|_{v=0} &= 0,\\
\boldsymbol{\Sigma}|_{v=0} &= \boldsymbol{\sigma},\\[2ex]
[\dot{\Sigma}^0]\Big|_{v=0} &= (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}),\\
[\dot{\boldsymbol{\Sigma}}]\Big|_{v=0} &= \dot{\boldsymbol{\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
[\ddot{\Sigma}^0]\Big|_{v=0} &= 2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + (\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}), \\
[\ddot{\boldsymbol{\Sigma}}]\Big|_{v=0} &= \ddot{\boldsymbol{\sigma}} + (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
[\dddot{\Sigma}^0]\Big|_{v=0} &= 3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) + 3(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + (\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 4\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}), \\
[\dddot{\boldsymbol{\Sigma}}]\Big|_{v=0} &= \dddot{\boldsymbol{\sigma}} + \dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} + \frac{3}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + 3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

G.4.8 Covariant kinematical derivatives

Computing now the *covariant* proper-time derivatives of U and Σ, related to their partial counterparts by

$$
(\dot{C})^\alpha \equiv [\dot{C}^\alpha] + U^\alpha(C\cdot\dot{U}),
$$

we find

$$
\begin{aligned}
(\dot{U})^0 &= \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}), \\
(\dot{\boldsymbol{U}}) &= \gamma^2\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\ddot{U})^0 &= \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}) + 3\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2, \\
(\ddot{\boldsymbol{U}}) &= \gamma^3\ddot{\boldsymbol{v}} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 3\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\dddot{U})^0 &= \gamma^6(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}}) + 18\gamma^{10}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3 + 3\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}) + 10\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}), \\
(\dddot{\boldsymbol{U}}) &= \gamma^4\dddot{\boldsymbol{v}} + 3\gamma^6\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + 6\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + 4\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \gamma^6(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 18\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + 18\gamma^{10}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3\boldsymbol{v} + 3\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} \\
&\quad + 10\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\ddddot{U})^0 &= \gamma^7(\boldsymbol{v}\cdot\ddddot{\boldsymbol{v}}) + 10\gamma^9(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})^2 + 162\gamma^{13}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^4 + 9\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}) \\
&\quad + 57\gamma^{11}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2 + 15\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}}) + 10\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})
\end{aligned}
$$

$$
\begin{aligned}
&+124\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}),\\
(\dddot{\boldsymbol{U}}) &= \gamma^5\dddot{\boldsymbol{v}} + 9\gamma^7\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + 10\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + 10\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + 5\gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+ \gamma^7(\boldsymbol{v}\cdot\ddddot{\boldsymbol{v}})\boldsymbol{v} + 10\gamma^7(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 54\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 10\gamma^9(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{v}\\
&+ 162\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + 162\gamma^{13}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^4\boldsymbol{v} + 57\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+ 9\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 57\gamma^{11}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} + 70\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+ 15\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})\boldsymbol{v} + 10\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 124\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\dot{\Sigma})^0 &= \gamma^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) + \gamma^2(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - \gamma^4(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&+ \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}),\\
(\dot{\boldsymbol{\Sigma}}) &= \gamma\dot{\boldsymbol{\sigma}} + \gamma^3(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \gamma^3(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&- \gamma^4(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + \gamma^5(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\ddot{\Sigma})^0 &= \gamma^3(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}) + 2\gamma^3(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + \gamma^3(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&- 2\gamma^5(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) - \gamma^5(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + \gamma^5(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\\
&- \gamma^7(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) - 7\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&+ \gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 3\gamma^5(\gamma+1)(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&- 2\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) + 2\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\\
&+ 4\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}) - 2\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}),\\
(\ddot{\boldsymbol{\Sigma}}) &= \gamma^2\ddot{\boldsymbol{\sigma}} + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \gamma^4(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&+ 2\gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v}\\
&- 2\gamma^5(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} - \gamma^5(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&- \gamma^6(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \gamma^7(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} + \gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+ \gamma^6(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 4\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+ 3\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} - 2\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&- 2\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} - 5\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&+ 2\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 4\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}\\
&- 2\gamma^9(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\dddot{\Sigma})^0 \;&= \gamma^4(\boldsymbol{v}\cdot\dddot{\boldsymbol{\sigma}}) + 3\gamma^4(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) + 3\gamma^4(\gamma+1)^{-2}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ \gamma^4(\gamma+1)^{-2}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 3\gamma^5(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) + 3\gamma^5(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ \gamma^5(\gamma+1)^{-3}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - 6\gamma^7(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) - 6\gamma^7(\gamma+1)^{-2}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&- 2\gamma^7(\gamma+1)^{-2}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 3\gamma^8(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}) + 3\gamma^8(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ \gamma^8(\gamma+1)^{-3}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - 11\gamma^7\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 4\gamma^6(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 4\gamma^6(\gamma+1)\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 2\gamma^7(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 6\gamma^8(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - 2\gamma^9(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 6\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}) + 4\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) - 27\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&- 14\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 19\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) - 50\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \\
&+ 4\gamma^6(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 3\gamma^6(\gamma+1)^{-2}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) \\
&+ \gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 12\gamma^6(\gamma+1)(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 6\gamma^6(\gamma+1)(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 3\gamma^7(\gamma+1)^{-3}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \\
&- 3\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}) - 3\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&- 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + 6\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 3\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 19\gamma^8(\gamma+1)(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \\
&- 5\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - 21\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 3\gamma^{10}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 6\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}) \\
&+ 27\gamma^{10}(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) + 28\gamma^{10}(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}) \\
&- 6\gamma^{11}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) - 24\gamma^{11}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}) \\
&+ 6\gamma^{12}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 13\gamma^8\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) \\
&- 28\gamma^9(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) + 9\gamma^{10}(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) \\
&+ 13\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}) \\
&- 6\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}), \\
(\dddot{\boldsymbol{\Sigma}}) \;&= \gamma^3\dddot{\boldsymbol{\sigma}} + \gamma^5\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} \\
&+ 3\gamma^5(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \gamma^5(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\gamma^5(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} \\
&+ 3\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} + 3\gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \gamma^5(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{v} + 3\gamma^6(\gamma+1)^{-4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \gamma^6(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3\gamma^6(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 3\gamma^6(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} - \gamma^6(\gamma+1)^{-1}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&- 3\gamma^7(\gamma+1)^{-2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \gamma^8(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&- 6\gamma^8(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \gamma^8(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ 3\gamma^9(\gamma+1)^{-4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \gamma^7(\gamma+1)^{-4}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&+ 4\gamma^7(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 3\gamma^8(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 4\gamma^8(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 3\gamma^9(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&+ 3\gamma^9(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + \gamma^{10}(\gamma+1)^{-4}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&+ 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} + \gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 2\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 9\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ \gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 8\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ 15\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + 6\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&+ 5\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 4\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 6\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v} - 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} - 3\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} \\
&- 21\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 18\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 9\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 7\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 6\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 6\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&+ 3\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 3\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 28\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 19\gamma^9(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 24\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 21\gamma^{10}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v}
\end{aligned}
$$

$$
\begin{aligned}
&- 40\gamma^{10}(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ 6\gamma^{11}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} + 27\gamma^{11}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 28\gamma^{11}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 6\gamma^{12}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&- 24\gamma^{12}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 6\gamma^{13}(\gamma+1)^{-4}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 4\gamma^{9}(\gamma+1)^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 16\gamma^{10}(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 6\gamma^{11}(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&+ 13\gamma^{9}(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&- 6\gamma^{10}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v},
\end{aligned}
$$

Evaluating these expressions for $\boldsymbol{v} = \boldsymbol{0}$, we find

$$
\begin{aligned}
(\dot{U})^0\Big|_{v=0} &= 0, \\
(\dot{\boldsymbol{U}})\Big|_{v=0} &= \dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
(\ddot{U})^0\Big|_{v=0} &= 0, \\
(\ddot{\boldsymbol{U}})\Big|_{v=0} &= \ddot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
(\dddot{U})^0\Big|_{v=0} &= 0, \\
(\dddot{\boldsymbol{U}})\Big|_{v=0} &= \dddot{\boldsymbol{v}} + 3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
(\ddddot{U})^0\Big|_{v=0} &= 0, \\
(\ddddot{\boldsymbol{U}})\Big|_{v=0} &= \ddddot{\boldsymbol{v}} + 9\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + 10(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
(\dot{\Sigma})^0\Big|_{v=0} &= 0, \\
(\dot{\boldsymbol{\Sigma}})\Big|_{v=0} &= \dot{\boldsymbol{\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
(\ddot{\Sigma})^0\Big|_{v=0} &= 0, \\
(\ddot{\boldsymbol{\Sigma}})\Big|_{v=0} &= \ddot{\boldsymbol{\sigma}},
\end{aligned}
$$

$$(\dddot{\varSigma})^0\Big|_{v=0} = 0,$$
$$(\dddot{\boldsymbol{\Sigma}})\Big|_{v=0} = \ddot{\boldsymbol{\sigma}} + \dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} + \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{1}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

G.4.9 FitzGerald spin derivatives

The spin derivatives computed in Sections G.4.7 and G.4.8 were given in terms of the three-spin $\boldsymbol{\sigma}$; this vector is intuitively understandable, and has been universally used historically. However, as noted in Section G.4.5, some theoretical results are considerably simplified algbraically if rewritten in terms of the FitzGerald three-spin $\boldsymbol{\sigma'}$. We now recompute the partial and covariant derivatives of $\varSigma$ in terms of $\boldsymbol{\sigma'}$.

It may be shown that

$$\boldsymbol{\sigma} = \boldsymbol{\sigma'} + \gamma^2(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma'})\boldsymbol{v};$$

to verify this, simply substitute $\boldsymbol{\sigma'}$ into the right-hand side; one finds

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}.$$

Using $\boldsymbol{\sigma'}$ in the definition of $\varSigma$, and then differentiating the resulting expressions anew, both partially and covariantly, we find

$$\varSigma^0 = \gamma^2(\boldsymbol{v}\cdot\boldsymbol{\sigma'}),$$
$$\boldsymbol{\Sigma} = \boldsymbol{\sigma'} + \gamma^2(\boldsymbol{v}\cdot\boldsymbol{\sigma'})\boldsymbol{v},$$

$$[\dot{\varSigma}^0] = \gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma'}) + 2\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma'}),$$
$$[\dot{\boldsymbol{\Sigma}}] = \gamma\dot{\boldsymbol{\sigma}}' + \gamma^3(\boldsymbol{v}\cdot\boldsymbol{\sigma'})\dot{\boldsymbol{v}} + \gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + \gamma^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma'})\boldsymbol{v} + 2\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma'})\boldsymbol{v},$$

$$\begin{aligned}[\ddot{\varSigma}^0] = {} & \gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}') + 2\gamma^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma'}) + 2\gamma^6\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma'}) \\ & + 5\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 5\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma'}) + 2\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma'}) \\ & + 10\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma'}),\end{aligned}$$

$$
\begin{aligned}
[\ddot{\boldsymbol{\Sigma}}] \;=\; & \gamma^2\ddot{\boldsymbol{\sigma}}' + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}' + \gamma^4(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 2\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v} \\
& + 2\gamma^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 2\gamma^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + \gamma^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 2\gamma^6\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 5\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 5\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 5\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 2\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 10\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
[\dddot{\Sigma}^0] \;=\; & 3\gamma^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}') + 3\gamma^5(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^5(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + 7\gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 7\gamma^7\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') \\
& + 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}') + 18\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}') + 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') \\
& + 7\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 7\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + 2\gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}') \\
& + 6\gamma^7(\boldsymbol{v}\cdot\boldsymbol{\sigma}')(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + 40\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 40\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') \\
& + 80\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}') + 32\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}') \\
& + 32\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),
\end{aligned}
$$

$$
\begin{aligned}
[\dddot{\boldsymbol{\Sigma}}] \;=\; & \gamma^5\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}' + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}' + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}' + \gamma^5(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dddot{\boldsymbol{v}} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\ddot{\boldsymbol{v}} \\
& + 3\gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + 3\gamma^5(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 6\gamma^5(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + 3\gamma^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v} \\
& + 3\gamma^5(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 3\gamma^5(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + \gamma^5(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}' \\
& + 7\gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 7\gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 7\gamma^7\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 18\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v} \\
& + 18\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 18\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 7\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 7\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 7\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 2\gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 6\gamma^7(\boldsymbol{v}\cdot\boldsymbol{\sigma}')(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 40\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} \\
& + 40\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 40\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 80\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\
& + 32\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 32\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v},
\end{aligned}
$$

$$
\begin{aligned}
(\dot{\Sigma})^0 \;=\; & \gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}'), \\
(\dot{\boldsymbol{\Sigma}}) \;=\; & \gamma\dot{\boldsymbol{\sigma}}' + \gamma^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + \gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + \gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v},
\end{aligned}
$$

$$
(\ddot{\Sigma})^0 \;=\; \gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}') + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + \gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')
$$

$$+ 4\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),$$

$$(\ddot{\boldsymbol{\Sigma}}) = \gamma^2\ddot{\boldsymbol{\sigma}}' + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}' + \gamma^4(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 2\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + \gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v}$$
$$+ \gamma^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 4\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v}$$
$$+ \gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + \gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 4\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v},$$

$$(\dddot{\Sigma})^0 = \gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 6\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}') + 3\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}') + \gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')$$
$$+ 4\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 2\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + \gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')$$
$$+ 19\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + 9\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + 28\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')$$
$$+ 4\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}') + 13\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),$$

$$(\dddot{\boldsymbol{\Sigma}}) = \gamma^5\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}' + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}' + \gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}' + \gamma^5(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dddot{\boldsymbol{v}} + 3\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\ddot{\boldsymbol{v}}$$
$$+ 3\gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + 2\gamma^5(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 3\gamma^5(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + \gamma^5(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}}$$
$$+ 4\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}' + 4\gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + \gamma^7\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v}$$
$$+ 8\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 15\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} + 6\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v}$$
$$+ 9\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 3\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + \gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v}$$
$$+ 5\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 4\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} + 2\gamma^7(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v}$$
$$+ \gamma^7(\boldsymbol{v}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 28\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 19\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v}$$
$$+ 9\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 28\gamma^{11}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} + 4\gamma^9\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v}$$
$$+ 13\gamma^9(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v},$$

G.4.10 Electric parts of six-vectors

We now obtain explicit expressions for the electric parts of the six-vectors formed from the various four-vectors U, $[\dot{U}]$, $[\ddot{U}]$, Σ, $[\dot{\Sigma}]$, $[\ddot{\Sigma}]$ and ζ—where $\zeta \equiv (R, R\boldsymbol{n})$,—that are required in order to compute the retarded electric fields from electric charges and electric dipole moments. We find

$$U^0\boldsymbol{\Sigma} - \Sigma^0\boldsymbol{U} = \gamma\boldsymbol{\sigma}',$$

$$U^0\dot{\boldsymbol{U}} - \dot{U}^0\boldsymbol{U} = \gamma^3\dot{\boldsymbol{v}},$$

$$\Sigma^0\dot{\boldsymbol{U}} - \dot{U}^0\boldsymbol{\Sigma} = -\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}' + \gamma^4(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}},$$

$$U^0\boldsymbol{\zeta} - \zeta^0\boldsymbol{U} = -R\gamma\boldsymbol{v} + R\gamma\boldsymbol{n},$$

$$\zeta^0\dot{\boldsymbol{U}} - \dot{U}^0\boldsymbol{\zeta} = R\gamma^2\dot{\boldsymbol{v}} + R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} - R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{n},$$

$$\begin{aligned}\zeta^0\ddot{\boldsymbol{U}} - \ddot{U}^0\boldsymbol{\zeta} = {} & R\gamma^3\ddot{\boldsymbol{v}} + R\gamma^5\dot{\boldsymbol{v}}^2\boldsymbol{v} - R\gamma^5\dot{\boldsymbol{v}}^2\boldsymbol{n} + 3R\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ & + R\gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} - R\gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n} + 4R\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{v} \\ & - 4R\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\zeta^0\dot{\boldsymbol{\Sigma}} - \dot{\Sigma}^0\boldsymbol{\zeta} = {} & R\gamma\dot{\boldsymbol{\sigma}}' + R\gamma^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + R\gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} - R\gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n} \\ & + R\gamma^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} - R\gamma^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{n} + 2R\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\ & - 2R\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\zeta^0\ddot{\boldsymbol{\Sigma}} - \ddot{\Sigma}^0\boldsymbol{\zeta} = {} & R\gamma^2\ddot{\boldsymbol{\sigma}}' + R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}' + R\gamma^4(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\ddot{\boldsymbol{v}} + 2R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\dot{\boldsymbol{v}} \\ & + R\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{v} - R\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{n} + 2R\gamma^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} \\ & + 2R\gamma^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} - 2R\gamma^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n} + R\gamma^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\ & - R\gamma^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{n} + 2R\gamma^6\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} - 2R\gamma^6\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{n} \\ & + 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\dot{\boldsymbol{v}} + 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{v} \\ & - 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n} + 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\ & - 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{n} + 2R\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\ & - 2R\gamma^6(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{n} + 10R\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{v} \\ & - 10R\gamma^8(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\boldsymbol{n}.\end{aligned}$$

G.4.11 Dot-products of four-vectors

Finally, we compute the various dot-products of the four-vectors U, $[\dot{U}]$, $[\ddot{U}]$, Σ, $[\dot{\Sigma}]$, $[\ddot{\Sigma}]$ and ζ that are required to compute the retarded electromagnetic fields. These products are labelled as follows:

$$\begin{aligned}
\varphi &\equiv (\zeta\cdot U)^{-1},\\
\psi &\equiv (\zeta\cdot \Sigma),\\
\dot\chi &\equiv (\zeta\cdot \dot U),\\
\dot\vartheta &\equiv [U\cdot \dot\Sigma],\\
\dot\psi &\equiv [\zeta\cdot \dot\Sigma],\\
\ddot\chi &\equiv [\zeta\cdot \ddot U],\\
\ddot\eta &\equiv [\zeta\cdot \ddot\Sigma].
\end{aligned}$$

One finds

$$\varphi^{-1} = R\gamma - R\gamma(\boldsymbol{v}\cdot\boldsymbol{n}),$$

$$\psi = -R(\boldsymbol{\sigma}'\cdot\boldsymbol{n}) + \varphi^{-1}\gamma(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),$$

$$\dot\chi = -R\gamma^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}) + \varphi^{-1}\gamma^3(\boldsymbol{v}\cdot\dot{\boldsymbol{v}}),$$

$$\dot\vartheta = \gamma^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + \gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),$$

$$\begin{aligned}
\dot\psi = {}& -R\gamma(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}) + \varphi^{-1}\gamma^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}') + \varphi^{-1}\gamma^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') - R\gamma^3(\boldsymbol{v}\cdot\boldsymbol{\sigma}')(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\\
& + 2\varphi^{-1}\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}'),
\end{aligned}$$

$$\begin{aligned}
\ddot\chi = {}& \varphi^{-1}\gamma^4\dot{\boldsymbol{v}}^2 - R\gamma^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n}) + \varphi^{-1}\gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}}) + 4\varphi^{-1}\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\\
& - 3R\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}),
\end{aligned}$$

$$\begin{aligned}
\ddot\eta = {}& -R\gamma^2(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}) + \varphi^{-1}\gamma^3(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}}') + 2\varphi^{-1}\gamma^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}') + \varphi^{-1}\gamma^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\\
& + 2\varphi^{-1}\gamma^5\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}') - R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}) - R\gamma^4(\boldsymbol{v}\cdot\boldsymbol{\sigma}')(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\\
& - 2R\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}) - 2R\gamma^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}) + 5\varphi^{-1}\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\\
& + 5\varphi^{-1}\gamma^5(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}') + 2\varphi^{-1}\gamma^5(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}') + 10\varphi^{-1}\gamma^7(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma}')\\
& - 5R\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma}')(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}).
\end{aligned}$$

G.4.12 Radiation reaction torque

We now compute just one consequence of the radiation reaction results of Chapter 6, namely, the finite radiation reaction torque on a charged magnetic dipole, due to its Thomas–BMT motion. It was found in Chapter 6 that the finite terms in the torque expression, for $\boldsymbol{v}{=}\boldsymbol{0}$, are

$$\boldsymbol{N}_{\text{self}} = \frac{1}{3}\mu^2\eta_0\,\boldsymbol{\sigma}\times\Big\{2\ddot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\times(\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}})\Big\}. \tag{G.10}$$

From the expressions in Section G.4.8, one finds that (G.10) can be written in covariant form as

$$(\dot{S}) = \frac{2}{3}\mu^2\eta_0\,U\times\Sigma\times\Big\{(\dddot{\Sigma}) + \dot{U}^2(\dot{\Sigma})\Big\}. \tag{G.11}$$

From (G.11), one can find the rate of change of the three-spin $\boldsymbol{\sigma}$, in any lab frame:

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\times\left\{\frac{1}{\gamma}\boldsymbol{C} - \frac{1}{\gamma+1}C^0\boldsymbol{v}\right\} + \dot{\boldsymbol{\sigma}}_T \equiv \boldsymbol{\sigma}\times\boldsymbol{\Omega}_{\text{RR}} + \dot{\boldsymbol{\sigma}}_T, \tag{G.12}$$

where $\dot{\boldsymbol{\sigma}}_T$ is the Thomas precession contribution to $\dot{\boldsymbol{\sigma}}$, and

$$C \equiv \frac{2}{3}\frac{\mu^2}{s}\eta_0\Big\{(\dddot{\Sigma}) + \dot{U}^2(\dot{\Sigma})\Big\}.$$

We remove inconvenient constants in $\boldsymbol{\Omega}_{\text{RR}}$ by defining a related vector $\boldsymbol{\Omega}'_{\text{RR}}$:

$$\boldsymbol{\Omega}_{\text{RR}} \equiv \frac{\mu^2}{6\pi s}\boldsymbol{\Omega}'_{\text{RR}}. \tag{G.13}$$

Apart from the Thomas precession contribution, we use (G.11) and (G.13), and the expressions in Section G.4.8, to find the general rate of change of spin in any lab frame:

$$\begin{aligned}\boldsymbol{\Omega}'_{\text{RR}} &= \gamma^2\dddot{\boldsymbol{\sigma}} + 3\gamma^4(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \gamma^4(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + 3\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}\\ &\quad + 3\gamma^4(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \gamma^4(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\ &\quad + 2\gamma^4(\gamma+1)^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}\end{aligned}$$

$$
\begin{aligned}
&+ 3\gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} + 3\gamma^4(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- 3\gamma^4(\gamma+1)^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{v} - 3\gamma^4(\gamma+1)^{-1}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- \gamma^4(\gamma+1)^{-1}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 3\gamma^5(\gamma+1)^{-4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \gamma^5(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3\gamma^6(\gamma+1)^{-2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \gamma^7(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 6\gamma^7(\gamma+1)^{-3}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \gamma^7(\gamma+1)^{-3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3\gamma^8(\gamma+1)^{-4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ 3\gamma^6(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 3\gamma^6(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 3\gamma^6(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 9\gamma^6(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 3\gamma^6(\gamma+1)^{-2}(\boldsymbol{v}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{v} + 8\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ 15\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - 9\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{v} \\
&- 5\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 5\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 2\gamma^6(\gamma+1)^{-1}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} - 3\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&- 6\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - 3\gamma^7(\gamma+1)^{-2}(\boldsymbol{v}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 21\gamma^7(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 6\gamma^8(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ 27\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 9\gamma^8(\gamma+1)^{-1}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{v} \\
&- 24\gamma^9(\gamma+1)^{-2}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 6\gamma^{10}(\gamma+1)^{-3}(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{v}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 9\gamma^8(\gamma+1)^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v} + 3\gamma^9(\gamma+1)^{-3}\dot{\boldsymbol{v}}^2(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\cdot\boldsymbol{\sigma})\boldsymbol{v}.
\end{aligned}
$$

When $\boldsymbol{v}=\boldsymbol{0}$, we find

$$
\boldsymbol{\Omega}'_{\mathrm{RR}} = \ddot{\boldsymbol{\sigma}} + \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{1}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
$$

in agreement with the original expression (G.10).

For circular motion and uniform precession around the same axis, with rates Ω_M and Ω_P respectively, we construct Cartesian axes with $\boldsymbol{i}$ in the direction of the frequencies' axis, and $\boldsymbol{k}$ in the direction of $\boldsymbol{v}$, and parametrise the spin direction $\boldsymbol{\sigma}$ according to

$$
\boldsymbol{\sigma} \equiv \boldsymbol{i}\sin\theta - \boldsymbol{j}\cos\theta\sin\phi + \boldsymbol{k}\cos\theta\cos\phi,
$$

where $\phi(t) \equiv \mathit{\Omega}_P t$. We then find

$$\begin{aligned}
\boldsymbol{\Omega}'_{\mathrm{RR}} = {} & \mathit{\Omega}_P^3\gamma^2\boldsymbol{\sigma}\times\boldsymbol{i} - 8\mathit{\Omega}_M^2\mathit{\Omega}_P\gamma^4\boldsymbol{\sigma}\times\boldsymbol{i} + 2\mathit{\Omega}_M^3\gamma^3(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} \\
& - 4\mathit{\Omega}_M^3\gamma^5(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} + 2\mathit{\Omega}_M^3\gamma^7(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} \\
& - 3\mathit{\Omega}_M\mathit{\Omega}_P^2\gamma^2(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} - 3\mathit{\Omega}_M\mathit{\Omega}_P^2\gamma^3(\gamma+1)^{-3}\boldsymbol{\sigma}\times\boldsymbol{i} \\
& + 6\mathit{\Omega}_M\mathit{\Omega}_P^2\gamma^5(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} - 3\mathit{\Omega}_M\mathit{\Omega}_P^2\gamma^6(\gamma+1)^{-3}\boldsymbol{\sigma}\times\boldsymbol{i} \\
& + 2\mathit{\Omega}_M^2\mathit{\Omega}_P\gamma^2(\gamma+1)^{-3}\boldsymbol{\sigma}\times\boldsymbol{i} + 20\mathit{\Omega}_M^2\mathit{\Omega}_P\gamma^5(\gamma+1)^{-1}\boldsymbol{\sigma}\times\boldsymbol{i} \\
& - 10\mathit{\Omega}_M^2\mathit{\Omega}_P\gamma^6(\gamma+1)^{-2}\boldsymbol{\sigma}\times\boldsymbol{i} + 2\mathit{\Omega}_M^2\mathit{\Omega}_P\gamma^7(\gamma+1)^{-3}\boldsymbol{\sigma}\times\boldsymbol{i}\,.
\end{aligned}$$

Substituting the Lorentz–Thomas result of

$$\mathit{\Omega}_P = (1 + a\gamma)\mathit{\Omega}_M,$$

where

$$a \equiv \frac{g-2}{2}$$

is the magnetic anomaly, we find

$$\begin{aligned}
\boldsymbol{\Omega}'_{\mathrm{RR}} = {} & -\mathit{\Omega}_M^3\gamma^3\boldsymbol{\sigma}\times\boldsymbol{i} + 2\mathit{\Omega}_M^3\gamma^5\boldsymbol{\sigma}\times\boldsymbol{i} - \mathit{\Omega}_M^3 a\gamma^3\boldsymbol{\sigma}\times\boldsymbol{i} + 4\mathit{\Omega}_M^3 a\gamma^5\boldsymbol{\sigma}\times\boldsymbol{i} \\
& + 3\mathit{\Omega}_M^3 a^2\gamma^5\boldsymbol{\sigma}\times\boldsymbol{i} + \mathit{\Omega}_M^3 a^3\gamma^5\boldsymbol{\sigma}\times\boldsymbol{i}\,.
\end{aligned}$$

This result is discussed in Chapter 6.

G.5 RETFIELD: Retarded fields

G.5.1 Introduction

This program verifies that the explicit expressions obtained by the author for the retarded fields agree with those extractable directly from the manifestly-covariant expressions.

G.5.2 Covariant field expressions

Since a significant amount of intelligence is required to convert the manifestly-covariant retarded field expressions to their final simplified non-covariant form, we will here only verify that the initial and final expressions agree, when all convenient quantities are expanded out explicitly. We start with the manifestly covariant field expressions which, in the case of the dipole fields, have been explicitly verified against the Cohn–Wiebe expressions:

$$
\begin{aligned}
F_2^q &= \varphi^3 \zeta \wedge U, \\
F_1^q &= \varphi^2 \zeta \wedge \dot{U} - \varphi^3 \dot{\chi} \zeta \wedge U, \\
F_3^d &= \varphi^3 U \wedge \Sigma - 3\varphi^5 \psi \zeta \wedge U, \\
F_2^d &= \varphi^2 \dot{U} \wedge \Sigma + \varphi^3 [\zeta \wedge \dot{\Sigma}] + \varphi^3 \psi U \wedge \dot{U} - \varphi^3 \dot{\chi} U \wedge \Sigma \\
&\quad + 6\varphi^5 \dot{\chi} \psi \zeta \wedge U - 3\varphi^4 \dot{\psi} \zeta \wedge U - 3\varphi^4 \psi \zeta \wedge \dot{U} + \varphi^3 \dot{\vartheta} \zeta \wedge U, \\
F_1^d &= \varphi^2 [\zeta \wedge \ddot{\Sigma}] - \varphi^3 \psi [\zeta \wedge \ddot{U}] + \varphi^4 \psi \ddot{\chi} \zeta \wedge U - 2\varphi^3 \dot{\psi} \zeta \wedge \dot{U} + 3\varphi^4 \dot{\chi} \dot{\psi} \zeta \wedge U \\
&\quad - \varphi^3 \dot{\chi} [\zeta \wedge \dot{\Sigma}] - \varphi^3 \ddot{\eta} \zeta \wedge U - 3\varphi^5 \dot{\chi}^2 \psi \zeta \wedge U + 3\varphi^4 \psi \dot{\chi} \zeta \wedge \dot{U},
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta &\equiv x - z, \\
\varphi &\equiv (\zeta \cdot U)^{-1}, \\
\dot{\chi} &\equiv (\zeta \cdot \dot{U}), \\
\ddot{\chi} &\equiv [\zeta \cdot \ddot{U}], \\
\psi &\equiv (\zeta \cdot \Sigma), \\
\dot{\psi} &\equiv [\zeta \cdot \dot{\Sigma}], \\
\dot{\vartheta} &\equiv [U \cdot \dot{\Sigma}], \\
\ddot{\eta} &\equiv [\zeta \cdot \ddot{\Sigma}].
\end{aligned}
$$

G.5.3 Directly extracted expressions

Using the expressions calculated in the program KINEMATS in the above expressions, one can extract the following explicit expressions for the retarded fields, by direct calculation:

$$\boldsymbol{E}_1'^q = -\kappa^2\dot{\boldsymbol{v}} - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n},$$

$$\boldsymbol{E}_2'^q = -\kappa^3\gamma^{-2}\boldsymbol{v} + \kappa^3\gamma^{-2}\boldsymbol{n},$$

$$\begin{aligned}\boldsymbol{E}_1'^d = {} & -\kappa^2\ddot{\boldsymbol{\sigma}}' - \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\ddot{\boldsymbol{v}} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} - \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v} + \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{n} \\ & - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{\sigma}}' - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{v}} - \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} \\ & + \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} - 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} \\ & - 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{v} + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_2'^d = {} & -\kappa^3\gamma^{-2}\dot{\boldsymbol{\sigma}}' + 2\kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\dot{\boldsymbol{v}} + \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{v} - \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{n} + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} \\ & + 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{v} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{n} - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}' - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} \\ & - 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v} + 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} \\ & - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{n} \\ & - 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_3'^d = {} & -\kappa^3\gamma^{-2}\boldsymbol{\sigma}' + 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{v} - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{n} - 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v} \\ & + 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{n},\end{aligned}$$

$$\boldsymbol{B}_1'^q = \kappa^2\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n},$$

$$\boldsymbol{B}_2'^q = \kappa^3\gamma^{-2}\boldsymbol{v}\times\boldsymbol{n},$$

$$\begin{aligned}
\boldsymbol{B}_1'^d \;=\; & \kappa^2\ddot{\boldsymbol{\sigma}}'\times\boldsymbol{n} + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\ddot{\boldsymbol{v}}\times\boldsymbol{n} + 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\
& + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{\sigma}}'\times\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\
& + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{v}\times\boldsymbol{n},
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{B}_2'^d \;=\; & \kappa^2\boldsymbol{\sigma}'\times\dot{\boldsymbol{v}} + \kappa^3\gamma^{-2}\dot{\boldsymbol{\sigma}}'\times\boldsymbol{n} - 2\kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\dot{\boldsymbol{v}}\times\boldsymbol{n} - \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{v}\times\boldsymbol{n} \\
& + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v}\times\dot{\boldsymbol{v}} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{v}\times\boldsymbol{n} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}'\times\boldsymbol{v} \\
& + 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\
& - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}\times\boldsymbol{n} + 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n},
\end{aligned}$$

$$\boldsymbol{B}_3'^d \;=\; \kappa^3\gamma^{-2}\boldsymbol{\sigma}'\times\boldsymbol{v} - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{v}\times\boldsymbol{n} + 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n}.$$

G.5.4 The author's expressions

The final, simplified expressions obtained by the author (on paper) for the retarded fields are as follows:

$$\boldsymbol{E}_1'^q \;=\; \kappa^3\boldsymbol{n}\times\boldsymbol{E}_1''^q,$$

where

$$\boldsymbol{E}_1''^q \;=\; -\dot{\boldsymbol{v}}\times\boldsymbol{n}';$$

hence,

$$\boldsymbol{E}_1'^q \;=\; \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n}' - \kappa^3(\boldsymbol{n}\cdot\boldsymbol{n}')\dot{\boldsymbol{v}};$$

$$\boldsymbol{E}_2'^q \;=\; \kappa^3\gamma^{-2}\boldsymbol{n}',$$

$$\boldsymbol{E}_1'^d \;=\; \kappa^3\boldsymbol{n}\times\boldsymbol{E}_1''^d,$$

where

$$\begin{aligned}\boldsymbol{E}_1^{\prime\prime d} \;=\; & -\dot{\boldsymbol{\sigma}}'\times\dot{\boldsymbol{v}} - \ddot{\boldsymbol{\sigma}}'\times\boldsymbol{n}' - \kappa(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\ddot{\boldsymbol{v}}\times\boldsymbol{n}' - 3\kappa(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n}' \\ & - 3\kappa^2(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n}';\end{aligned}$$

hence,

$$\begin{aligned}\boldsymbol{E}_1^{\prime d} \;=\; & \kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} + \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{n}' - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{\sigma}}' - \kappa^3(\boldsymbol{n}\cdot\boldsymbol{n}')\ddot{\boldsymbol{\sigma}}' \\ & + \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n}' - \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}')\ddot{\boldsymbol{v}} + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n}' \\ & - 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}')\dot{\boldsymbol{v}} + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{n}' \\ & - 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\boldsymbol{n}')\dot{\boldsymbol{v}};\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_2^{\prime d} \;=\; & -\kappa^3\gamma^{-2}\dot{\boldsymbol{\sigma}}' - \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{n}' + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n}')\dot{\boldsymbol{v}} + \kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{n}' - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}' \\ & + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n}'')\boldsymbol{n}' + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n}'')\boldsymbol{n}' + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n}'')(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n}' \\ & - 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n}'')(\boldsymbol{n}\cdot\boldsymbol{n}')\dot{\boldsymbol{v}},\end{aligned}$$

$$\boldsymbol{E}_3^{\prime d} \;=\; -\kappa^3\gamma^{-2}\boldsymbol{\sigma}' + 3\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n}'')\boldsymbol{n}',$$

$$\boldsymbol{B}_1^{\prime q} \;=\; \boldsymbol{n}\times\boldsymbol{E}_1^{\prime q},$$

$$\boldsymbol{B}_2^{\prime q} \;=\; \boldsymbol{n}\times\boldsymbol{E}_2^{\prime q},$$

$$\boldsymbol{B}_1^{\prime d} \;=\; \boldsymbol{n}\times\boldsymbol{E}_1^{\prime d},$$

$$\boldsymbol{B}_2^{\prime d} \;=\; \boldsymbol{n}\times\boldsymbol{E}_2^{\prime d} + \kappa^2\boldsymbol{\sigma}'\times\dot{\boldsymbol{v}} + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n}' - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}'\times\boldsymbol{n}',$$

$$\boldsymbol{B}_3^{\prime d} \;=\; \boldsymbol{v}\times\boldsymbol{E}_3^{\prime d}.$$

Expanding out these expressions explicitly, we find

$$\boldsymbol{E}_1'^q = -\kappa^2\dot{\boldsymbol{v}} - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n},$$

$$\boldsymbol{E}_2'^q = -\kappa^3\gamma^{-2}\boldsymbol{v} + \kappa^3\gamma^{-2}\boldsymbol{n},$$

$$\begin{aligned}\boldsymbol{E}_1'^d = & -\kappa^2\ddot{\boldsymbol{\sigma}}' - \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\ddot{\boldsymbol{v}} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} - \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v} + \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{n} \\ & - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{\sigma}}' - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{v}} - \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} \\ & + \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} - 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} \\ & - 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{v} + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_2'^d = & -\kappa^3\gamma^{-2}\dot{\boldsymbol{\sigma}}' + 2\kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\dot{\boldsymbol{v}} + \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{v} - \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{n} + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} \\ & + 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{v} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{n} - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}' - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}} \\ & - 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v} + 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} \\ & - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v} - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{n} \\ & - 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v} + 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{n},\end{aligned}$$

$$\begin{aligned}\boldsymbol{E}_3'^d = & -\kappa^3\gamma^{-2}\boldsymbol{\sigma}' + 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{v} - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{n} - 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v} \\ & + 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{n},\end{aligned}$$

$$\boldsymbol{B}_1'^q = \kappa^2\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n},$$

$$\boldsymbol{B}_2'^q = \kappa^3\gamma^{-2}\boldsymbol{v}\times\boldsymbol{n},$$

$$\begin{aligned}\boldsymbol{B}_1'^d = & \kappa^2\ddot{\boldsymbol{\sigma}}'\times\boldsymbol{n} + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\ddot{\boldsymbol{v}}\times\boldsymbol{n} + 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^3(\ddot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\ & + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{\sigma}}'\times\boldsymbol{n} + 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + \kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\ & + 3\kappa^4(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} + 3\kappa^5(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})^2\boldsymbol{v}\times\boldsymbol{n},\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{B}_2'^d = {} & \kappa^2\boldsymbol{\sigma}'\times\dot{\boldsymbol{v}} + \kappa^3\gamma^{-2}\dot{\boldsymbol{\sigma}}'\times\boldsymbol{n} - 2\kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\dot{\boldsymbol{v}}\times\boldsymbol{n} - \kappa^3(\boldsymbol{\sigma}'\cdot\dot{\boldsymbol{v}})\boldsymbol{v}\times\boldsymbol{n} \\
& + \kappa^3(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v}\times\dot{\boldsymbol{v}} - 2\kappa^3(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{v})\boldsymbol{v}\times\boldsymbol{n} + \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{\sigma}'\times\boldsymbol{v} \\
& + 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\dot{\boldsymbol{v}}\times\boldsymbol{n} + 3\kappa^4\gamma^{-2}(\dot{\boldsymbol{\sigma}}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{v})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n} \\
& - 3\kappa^4(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\boldsymbol{v}\cdot\dot{\boldsymbol{v}})\boldsymbol{v}\times\boldsymbol{n} + 6\kappa^5\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})(\dot{\boldsymbol{v}}\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n},
\end{aligned}
$$

$$
\boldsymbol{B}_3'^d = \kappa^3\gamma^{-2}\boldsymbol{\sigma}'\times\boldsymbol{v} - 3\kappa^4\gamma^{-2}(\boldsymbol{\sigma}'\cdot\boldsymbol{v})\boldsymbol{v}\times\boldsymbol{n} + 3\kappa^5\gamma^{-4}(\boldsymbol{\sigma}'\cdot\boldsymbol{n})\boldsymbol{v}\times\boldsymbol{n}.
$$

G.5.5 Comparison of expressions

To compare these two sets of results quickly, we simply subtract the latter from the former; we find

$$
\begin{aligned}
\Delta\boldsymbol{E}_1'^q &= \boldsymbol{0}, \\
\Delta\boldsymbol{E}_2'^q &= \boldsymbol{0}, \\
\Delta\boldsymbol{E}_1'^d &= \boldsymbol{0}, \\
\Delta\boldsymbol{E}_2'^d &= \boldsymbol{0}, \\
\Delta\boldsymbol{E}_3'^d &= \boldsymbol{0}, \\
\Delta\boldsymbol{B}_1'^q &= \boldsymbol{0}, \\
\Delta\boldsymbol{B}_2'^q &= \boldsymbol{0}, \\
\Delta\boldsymbol{B}_1'^d &= \boldsymbol{0}, \\
\Delta\boldsymbol{B}_2'^d &= \boldsymbol{0}, \\
\Delta\boldsymbol{B}_3'^d &= \boldsymbol{0}.
\end{aligned}
$$

Hence, the author's simplified expressions for the retarded fields have been explicitly verified.

G.6 RADREACT: Radiation reaction

G.6.1 Introduction

This program computes the radiation reaction equations of motion for particles carrying electric charge and electric and magnetic dipole moments.

G.6.2 Pointlike particle trajectory

The following four-vector expression is an input to the program, computed by the author on paper:

$$\begin{aligned} t(\tau) \;&=\; \tau + \frac{1}{6}\tau^3\dot{\boldsymbol{v}}^2 + \frac{1}{8}\tau^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \frac{13}{120}\tau^5\dot{\boldsymbol{v}}^4 + \frac{1}{40}\tau^5\ddot{\boldsymbol{v}}^2 + \frac{1}{30}\tau^5(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) \\ &\quad + \frac{1}{144}\tau^6(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) + \frac{1}{72}\tau^6(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + \frac{3}{16}\tau^6\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \mathrm{O}(\tau^7), \\ \boldsymbol{z}(\tau) \;&=\; \frac{1}{2}\tau^2\dot{\boldsymbol{v}} + \frac{1}{6}\tau^3\ddot{\boldsymbol{v}} + \frac{1}{24}\tau^4\dddot{\boldsymbol{v}} + \frac{1}{6}\tau^4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + \frac{1}{120}\tau^5\ddddot{\boldsymbol{v}} + \frac{1}{12}\tau^5\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} \\ &\quad + \frac{1}{8}\tau^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \mathrm{O}(\tau^6), \end{aligned} \qquad \text{(G.14)}$$

Although these expressions are tedious to derive, they are easy to verify. Firstly, we compute $\dot{z}^2(\tau)$, by differentiating (G.14) with respect to τ:

$$\dot{z}^2(\tau) \;=\; 1,$$

the required result. We now compute $\boldsymbol{v}(\tau) \equiv d_\tau \boldsymbol{z}(\tau)/d_\tau t(\tau)$:

$$\begin{aligned} \boldsymbol{v}(\tau) \;&=\; \tau\dot{\boldsymbol{v}} + \frac{1}{2}\tau^2\ddot{\boldsymbol{v}} + \frac{1}{6}\tau^3\dddot{\boldsymbol{v}} + \frac{1}{6}\tau^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + \frac{1}{24}\tau^4\ddddot{\boldsymbol{v}} + \frac{1}{6}\tau^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + \frac{1}{8}\tau^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad + \mathrm{O}(\tau^5); \end{aligned} \qquad \text{(G.15)}$$

computing $\gamma(\tau) \equiv (1 - \boldsymbol{v}^2(\tau))^{-1/2}$, we find

$$\begin{aligned} \gamma(\tau) \;&=\; 1 + \frac{1}{2}\tau^2\dot{\boldsymbol{v}}^2 + \frac{1}{2}\tau^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \frac{13}{24}\tau^4\dot{\boldsymbol{v}}^4 + \frac{1}{8}\tau^4\ddot{\boldsymbol{v}}^2 + \frac{1}{6}\tau^4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) \\ &\quad + \frac{1}{24}\tau^5(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) + \frac{1}{12}\tau^5(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + \frac{9}{8}\tau^5\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \mathrm{O}(\tau^6). \end{aligned} \qquad \text{(G.16)}$$

Our second piece of verification evidence lies with the observation that $\gamma(\tau)$ may alternatively be computed via $\gamma(\tau) \equiv d_\tau t(\tau)$; this gives

$$\begin{aligned}\gamma(\tau) \;&=\; 1 + \frac{1}{2}\tau^2\dot{\boldsymbol{v}}^2 + \frac{1}{2}\tau^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \frac{13}{24}\tau^4\dot{\boldsymbol{v}}^4 + \frac{1}{8}\tau^4\ddot{\boldsymbol{v}}^2 + \frac{1}{6}\tau^4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) \\ &\quad + \frac{1}{24}\tau^5(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) + \frac{1}{12}\tau^5(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + \frac{9}{8}\tau^5\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \mathrm{O}(\tau^6),\end{aligned} \tag{G.17}$$

which is identical to (G.16). Now, upon reversion of $t(\tau)$, one finds

$$\begin{aligned}\tau(t) \;&=\; t - \frac{1}{6}t^3\dot{\boldsymbol{v}}^2 - \frac{1}{8}t^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{40}t^5\dot{\boldsymbol{v}}^4 - \frac{1}{40}t^5\ddot{\boldsymbol{v}}^2 - \frac{1}{30}t^5(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) \\ &\quad - \frac{1}{144}t^6(\dot{\boldsymbol{v}}\cdot\ddddot{\boldsymbol{v}}) - \frac{1}{72}t^6(\ddot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) - \frac{1}{24}t^6\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \mathrm{O}(t^7).\end{aligned} \tag{G.18}$$

Using (G.18) in (G.15), we thus find

$$\boldsymbol{v}(t) \;=\; t\dot{\boldsymbol{v}} + \frac{1}{2}t^2\ddot{\boldsymbol{v}} + \frac{1}{6}t^3\dddot{\boldsymbol{v}} + \frac{1}{24}t^4\ddddot{\boldsymbol{v}} + \mathrm{O}(t^5),$$

which of course *defines* $\dot{\boldsymbol{v}}$, $\ddot{\boldsymbol{v}}$, $\dddot{\boldsymbol{v}}$ and $\ddddot{\boldsymbol{v}}$; this completes the verification of the input expressions.

G.6.3 Trajectories of rigid body constituents

We now compute the trajectory of the constituent $\boldsymbol{r}$. This is given by

$$z_r^\alpha(\tau) = z^\alpha(\tau) + \Delta z_r^\alpha(\tau), \tag{G.19}$$

where

$$\begin{aligned}\Delta t_r(\tau) \;&=\; \gamma(\tau)\Big(\boldsymbol{u}(\tau)\cdot\boldsymbol{v}(\tau)\Big), \\ \Delta \boldsymbol{z}_r(\tau) \;&=\; \boldsymbol{u}(\tau) + \frac{\gamma^2(\tau)}{\gamma(\tau)+1}\Big(\boldsymbol{u}(\tau)\cdot\boldsymbol{v}(\tau)\Big)\boldsymbol{v}(\tau),\end{aligned} \tag{G.20}$$

where $\boldsymbol{u}(0) = \boldsymbol{r}$, and

$$d_\tau\boldsymbol{u}(\tau) = \frac{\gamma^3(\tau)}{\gamma(\tau)+1}\,\boldsymbol{u}(\tau)\times\Big(\boldsymbol{v}(\tau)\times\dot{\boldsymbol{v}}(\tau)\Big)\,. \tag{G.21}$$

Firstly, iterating the differential equation (G.21) for $\boldsymbol{u}(\tau)$, we find

$$\begin{aligned}\boldsymbol{u}(\tau) &= \boldsymbol{r} - \frac{1}{12}\tau^3(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{12}\tau^3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}\tau^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{24}\tau^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{1}{80}\tau^5(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddddot{\boldsymbol{v}} - \frac{1}{120}\tau^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{120}\tau^5(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{80}\tau^5(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{19}{240}\tau^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{19}{240}\tau^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \mathrm{O}(\tau^6). \qquad \text{(G.22)}\end{aligned}$$

As a cross-check of (G.22), we use (G.18) to replace τ by t, and then take the t-derivative of the result; we find

$$\begin{aligned}\dot{\boldsymbol{u}}(t) &= -\frac{1}{4}t^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{4}t^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{6}t^3(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{6}t^3(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{1}{16}t^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddddot{\boldsymbol{v}} - \frac{1}{24}t^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{24}t^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{16}t^4(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{3}{16}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{3}{16}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \mathrm{O}(\tau^5). \qquad \text{(G.23)}\end{aligned}$$

On the other hand, one may compute $\dot{\boldsymbol{u}}$ directly, via

$$\dot{\boldsymbol{u}}(t) = \frac{\gamma^2(t)}{\gamma(t)+1}\,\boldsymbol{u}(t)\times\Big(\boldsymbol{v}(t)\times\dot{\boldsymbol{v}}(t)\Big)\,;$$

one finds

$$\begin{aligned}\dot{\boldsymbol{u}}(t) &= -\frac{1}{4}t^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{4}t^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{6}t^3(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{6}t^3(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{1}{16}t^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddddot{\boldsymbol{v}} - \frac{1}{24}t^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{24}t^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{16}t^4(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ &\quad - \frac{3}{16}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{3}{16}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \mathrm{O}(\tau^5), \qquad \text{(G.24)}\end{aligned}$$

which is identical to (G.23).

Using the expression (G.22) in (G.20), we find

$$\begin{aligned}t_r(\tau) &= \tau + \tau(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}\tau^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{1}{6}\tau^3\dot{\boldsymbol{v}}^2 + \frac{1}{6}\tau^3(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}}) + \frac{2}{3}\tau^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) \\ &\quad + \frac{1}{24}\tau^4(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}}) + \frac{1}{8}\tau^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \frac{1}{2}\tau^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{13}{24}\tau^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) \\ &\quad + \mathrm{O}(\tau^5), \qquad \text{(G.25)}\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{z}_r(\tau) \;=\; & \boldsymbol{r} + \frac{1}{2}\tau^2\dot{\boldsymbol{v}} + \frac{1}{2}\tau^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{6}\tau^3\ddot{\boldsymbol{v}} + \frac{1}{6}\tau^3(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{3}\tau^3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& + \frac{1}{24}\tau^4\dddot{\boldsymbol{v}} + \frac{1}{6}\tau^4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + \frac{1}{24}\tau^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{8}\tau^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{8}\tau^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& + \frac{13}{24}\tau^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{120}\tau^5\ddddot{\boldsymbol{v}} + \frac{1}{12}\tau^5\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + \frac{1}{120}\tau^5(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddddot{\boldsymbol{v}} \\
& + \frac{1}{30}\tau^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{20}\tau^5(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{30}\tau^5(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}\tau^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& + \frac{7}{30}\tau^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{13}{30}\tau^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{11}{24}\tau^5(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& + \mathrm{O}(\tau^6). \qquad\qquad \text{(G.26)}
\end{aligned}
$$

As an explicit cross-check, we compute the four-scalar

$$\Delta z_r^2 \equiv \Delta t_r^2(\tau) - \Delta\boldsymbol{z}_r^2(\tau),$$

and find

$$\Delta z_r^2(\tau) \;=\; -\boldsymbol{r}^2,$$

as expected.

G.6.4 Rigid body redshift formula

We now compute $d_\tau\tau_r$, via

$$d_\tau\tau_r \equiv \sqrt{(d_\tau t_r)^2 - (d_\tau\boldsymbol{z}_r)^2};$$

from (G.25) and (G.26), the result is

$$
\begin{aligned}
d_\tau\tau_r \;=\; & 1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) + \tau(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{1}{2}\tau^2(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}}) + \frac{3}{2}\tau^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) + \frac{1}{6}\tau^3(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}}) \\
& + \frac{3}{2}\tau^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{5}{3}\tau^3(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \mathrm{O}(\tau^4).
\end{aligned}
$$

Of greatest importance,

$$d_\tau\tau_r\big|_{\tau=0} \;=\; 1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}});$$

we define the *redshift factor*

$$\lambda \equiv 1 + (\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) \tag{G.27}$$

for use in the following.

G.6.5 Lab-time constituent trajectories

We now obtain the trajectory of the constituent $\boldsymbol{r}$, *without* reference to the body τ at all. Reverting $t_r(\tau)$, we find

$$\begin{aligned}
\tau(t_r) \;=\; & \lambda^{-1}t_r - \frac{1}{2}\lambda^{-3}t_r^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}\lambda^{-4}t_r^3\dot{\boldsymbol{v}}^2 - \frac{1}{6}\lambda^{-4}t_r^3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) \\
& - \frac{2}{3}\lambda^{-4}t_r^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}\lambda^{-5}t_r^3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2 - \frac{1}{24}\lambda^{-5}t_r^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}}) \\
& - \frac{1}{8}\lambda^{-5}t_r^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{2}\lambda^{-5}t_r^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) - \frac{13}{24}\lambda^{-5}t_r^4(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) \\
& + \frac{5}{12}\lambda^{-6}t_r^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{5}{12}\lambda^{-6}t_r^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) + \frac{5}{3}\lambda^{-6}t_r^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}}) \\
& - \frac{5}{8}\lambda^{-7}t_r^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^3 + \mathrm{O}(t_r^5);
\end{aligned} \tag{G.28}$$

using (G.28) in (G.26) gives

$$\begin{aligned}
\boldsymbol{z}_r(t) \;=\; & \boldsymbol{r} + \frac{1}{2}\lambda^{-1}t^2\dot{\boldsymbol{v}} + \frac{1}{6}\lambda^{-2}t^3\ddot{\boldsymbol{v}} - \frac{1}{6}\lambda^{-3}t^3(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{24}\lambda^{-3}t^4\dddot{\boldsymbol{v}} \\
& - \frac{1}{8}\lambda^{-4}t^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{24}\lambda^{-4}t^4(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}\lambda^{-4}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& + \frac{1}{8}\lambda^{-5}t^4(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}\lambda^{-4}t^5\ddddot{\boldsymbol{v}} - \frac{1}{20}\lambda^{-5}t^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dddot{\boldsymbol{v}} \\
& - \frac{1}{30}\lambda^{-5}t^5(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{120}\lambda^{-5}t^5(\boldsymbol{r}\cdot\ddddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{10}\lambda^{-5}t^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& - \frac{1}{15}\lambda^{-5}t^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{12}\lambda^{-5}t^5(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}\lambda^{-6}t^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} \\
& + \frac{1}{12}\lambda^{-6}t^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}\lambda^{-6}t^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& - \frac{1}{8}\lambda^{-7}t^5(\boldsymbol{r}\cdot\ddot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(t^6),
\end{aligned} \tag{G.29}$$

where now t is understood to be t_r.

G.6.6 Infinitesimally small spherical bodies

We now consider rigid bodies that are infinitesimally small three-spheres, of radius ε, and compute the quantities required to calculate the self-fields. By taking $\boldsymbol{r}$ as well as t to be of order ε in (G.29), and relabelling $\boldsymbol{r}$ by $\boldsymbol{r}'$, we have

$$\begin{aligned}\boldsymbol{z}_{r'}(t) \;=&\; \boldsymbol{r}' + \frac{1}{2}t^2\dot{\boldsymbol{v}} + \frac{1}{6}t^3\ddot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{24}t^4\dddot{\boldsymbol{v}} - \frac{1}{3}t^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\ &- \frac{1}{6}t^3(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}t^5\ddddot{\boldsymbol{v}} - \frac{1}{8}t^4(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\\ &- \frac{1}{8}t^4(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{24}t^4(\boldsymbol{r}'\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}t^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\ &+ \frac{1}{2}t^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^6).\end{aligned} \qquad \text{(G.30)}$$

G.6.7 The retarded radius vector

The three-vector from the retarded constituent $\boldsymbol{r}'$ to the constituent $\boldsymbol{r}$ at $t = 0$ is denoted $\boldsymbol{R}$; hence,

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{z}_{r'}(t_{\text{ret}}), \qquad \text{(G.31)}$$

where by definition

$$t_{\text{ret}} \equiv -R. \qquad \text{(G.32)}$$

From (G.30), (G.31) and (G.32), we find

$$\begin{aligned}\boldsymbol{R} \;=&\; \boldsymbol{r} - \boldsymbol{r}' - \frac{1}{2}R^2\dot{\boldsymbol{v}} + \frac{1}{6}R^3\ddot{\boldsymbol{v}} + \frac{1}{2}R^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}R^4\dddot{\boldsymbol{v}} - \frac{1}{3}R^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\ &- \frac{1}{6}R^3(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}R^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}R^5\ddddot{\boldsymbol{v}} + \frac{1}{8}R^4(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\\ &+ \frac{1}{8}R^4(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{24}R^4(\boldsymbol{r}'\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}R^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\ &+ \frac{1}{2}R^3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}R^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^6).\end{aligned} \qquad \text{(G.33)}$$

G.6.8 Sum and difference variables

We now switch to the variables

$$\begin{aligned} \boldsymbol{r}_d &\equiv \boldsymbol{r} - \boldsymbol{r}', \\ \boldsymbol{r}_s &\equiv \boldsymbol{r} + \boldsymbol{r}', \end{aligned} \tag{G.34}$$

whence the reverse transformation is

$$\begin{aligned} \boldsymbol{r} &\equiv \frac{1}{2}\Big(\boldsymbol{r}_s + \boldsymbol{r}_d\Big), \\ \boldsymbol{r}' &\equiv \frac{1}{2}\Big(\boldsymbol{r}_s - \boldsymbol{r}_d\Big); \end{aligned} \tag{G.35}$$

using relations (G.35) in (G.33), we find

$$\begin{aligned} \boldsymbol{R} = \; & \boldsymbol{r}_d - \frac{1}{2}R^2\dot{\boldsymbol{v}} + \frac{1}{6}R^3\ddot{\boldsymbol{v}} - \frac{1}{4}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}R^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}R^4\dddot{\boldsymbol{v}} \\ & + \frac{1}{6}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{12}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{6}R^3(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{12}R^3(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ & - \frac{1}{8}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{1}{8}R^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{4}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{120}R^5\ddddot{\boldsymbol{v}} \\ & - \frac{1}{16}R^4(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - \frac{1}{16}R^4(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{48}R^4(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{16}R^4(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} \\ & + \frac{1}{16}R^4(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{48}R^4(\boldsymbol{r}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{16}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ & + \frac{1}{16}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + \frac{1}{8}R^3(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} \\ & + \frac{1}{8}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{4}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{8}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ & - \frac{1}{8}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}R^3(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{16}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} \\ & + \frac{1}{16}R^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} - \frac{3}{16}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{3}{16}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\ & + \mathrm{O}(\varepsilon^6). \end{aligned} \tag{G.36}$$

G.6.9 Final expression for retarded radius vector

Squaring (G.36), we find

$$\begin{aligned}
R^2 = {} & r_d^2 - R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) + \frac{1}{4}R^4\dot{\boldsymbol{v}}^2 + \frac{1}{3}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) - \frac{1}{2}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 \\
& + \frac{1}{2}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) - \frac{1}{6}R^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{12}R^4(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}}) + \frac{1}{4}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) \\
& - \frac{1}{4}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{1}{2}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) - \frac{1}{6}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) \\
& - \frac{1}{3}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) - \frac{1}{4}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^3 - \frac{1}{4}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 \\
& + \frac{1}{2}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{1}{36}R^6\ddot{\boldsymbol{v}}^2 + \frac{1}{24}R^6(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + \frac{1}{60}R^5(\boldsymbol{r}_d\cdot\ddddot{\boldsymbol{v}}) \\
& - \frac{1}{12}R^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) + \frac{1}{12}R^5\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) - \frac{1}{4}R^5(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) \\
& + \frac{1}{4}R^5(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{8}R^4(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})^2 + \frac{1}{16}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 \\
& + \frac{3}{16}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 - \frac{1}{6}R^4(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}}) + \frac{1}{24}R^4(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dddot{\boldsymbol{v}}) \\
& + \frac{1}{8}R^4(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) + \frac{1}{8}R^4(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) - \frac{1}{4}R^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\
& + \frac{1}{2}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) - \frac{1}{4}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) + \frac{1}{4}R^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 \\
& - \frac{3}{4}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{1}{4}R^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) - \frac{1}{8}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^4 \\
& + \frac{1}{8}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^3 - \frac{3}{8}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 + \frac{3}{8}R^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\
& + \mathrm{O}(\varepsilon^7);
\end{aligned} \tag{G.37}$$

substituting (G.37) into itself until R is eliminated, and then square-rooting, we find

$$\begin{aligned}
R = {} & r_d - \frac{1}{2}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) + \frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2 + \frac{1}{6}r_d^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) + \frac{1}{8}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 + \frac{1}{4}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\
& - \frac{1}{12}r_d^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) - \frac{1}{24}r_d^3(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}}) - \frac{3}{16}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}}) - \frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\
& - \frac{1}{12}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) - \frac{1}{12}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) - \frac{1}{6}r_d^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{16}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^3 - \frac{1}{8}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 - \frac{1}{8}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{7}{128}r_d^5\dot{\boldsymbol{v}}^4 \\
&+ \frac{1}{72}r_d^5\ddot{\boldsymbol{v}}^2 + \frac{1}{48}r_d^5(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}}) + \frac{1}{120}r_d^4(\boldsymbol{r}_d\cdot\ddddot{\boldsymbol{v}}) + \frac{1}{12}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) \\
&+ \frac{1}{24}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) + \frac{1}{8}r_d^4(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) + \frac{1}{8}r_d^4(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}}) \\
&+ \frac{1}{144}r_d^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})^2 + \frac{7}{64}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2 + \frac{3}{32}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 \\
&+ \frac{1}{48}r_d^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}}) + \frac{1}{48}r_d^3(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dddot{\boldsymbol{v}}) + \frac{1}{16}r_d^3(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) \\
&+ \frac{1}{16}r_d^3(\boldsymbol{r}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{11}{32}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \frac{1}{12}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}}) \\
&+ \frac{1}{24}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) + \frac{1}{8}r_d^2(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 + \frac{1}{8}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) \\
&+ \frac{1}{8}r_d^2(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\ddot{\boldsymbol{v}}) + \frac{3}{128}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^4 + \frac{1}{16}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^3 \\
&+ \frac{3}{32}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}})^2 + \frac{3}{32}r_d(\boldsymbol{r}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{r}_s\cdot\dot{\boldsymbol{v}}) + \mathrm{O}(\varepsilon^6). \qquad \text{(G.38)}
\end{aligned}
$$

Substituting (G.38) into (G.36), and switching to the variables r_d, r_s, $\boldsymbol{n}_d$ and $\boldsymbol{n}_s$, defined by

$$
\begin{aligned}
\boldsymbol{n}_d &\equiv \frac{\boldsymbol{r}_d}{r_d}, \\
\boldsymbol{n}_s &\equiv \frac{\boldsymbol{r}_s}{r_s}, \qquad \text{(G.39)}
\end{aligned}
$$

we find

$$
\begin{aligned}
\boldsymbol{R} = \; & r_d\boldsymbol{n}_d - \frac{1}{2}r_d^2\dot{\boldsymbol{v}} + \frac{1}{6}r_d^3\ddot{\boldsymbol{v}} + \frac{1}{4}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}r_d^2 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}r_d^4\dddot{\boldsymbol{v}} \\
&- \frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{8}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{1}{6}r_d^3 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{12}r_d^3 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{4}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{1}{8}r_d^2 r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{120}r_d^5\ddddot{\boldsymbol{v}} + \frac{1}{16}r_d^5\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + \frac{1}{48}r_d^5(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} \\
&+ \frac{1}{16}r_d^5(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + \frac{1}{16}r_d^5(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \frac{1}{48}r_d^5(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{48}r_d^5(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$+\frac{1}{12}r_d^5(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d^5\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{12}r_d^5(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{1}{16}r_d^4r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}+\frac{1}{16}r_d^4r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{48}r_d^4r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{1}{4}r_d^4r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d^4r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{12}r_d^4r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{3}{16}r_d^4r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d^4r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d^3r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$+\frac{3}{16}r_d^3r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}+\frac{1}{8}r_d^3r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{1}{16}r_d^2r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon^6). \tag{G.40}$$

G.6.10 The retarded normal vector

We now compute the vector $\boldsymbol{n}$, defined as

$$\boldsymbol{n}\equiv\frac{\boldsymbol{R}}{R}; \tag{G.41}$$

using (G.38), (G.39), (G.40) and (G.41), we find

$$\boldsymbol{n}=\boldsymbol{n}_d-\frac{1}{2}r_d\dot{\boldsymbol{v}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{6}r_d^2\ddot{\boldsymbol{v}}-\frac{1}{8}r_d^2\dot{\boldsymbol{v}}^2\boldsymbol{n}_d+\frac{1}{8}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$-\frac{1}{6}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{4}r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{4}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{24}r_d^3\dddot{\boldsymbol{v}}$$
$$-\frac{1}{16}r_d^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}-\frac{1}{16}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}+\frac{1}{16}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d+\frac{1}{24}r_d^3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{1}{12}r_d^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{16}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{12}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$-\frac{1}{6}r_d^2r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{1}{12}r_d^2r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d^2r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{1}{12}r_d^2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{8}r_d^2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{1}{6}r_d^2r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{8}r_dr_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}+\frac{1}{8}r_dr_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$+\frac{1}{120}r_d^4\ddddot{\boldsymbol{v}}+\frac{1}{24}r_d^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}-\frac{5}{128}r_d^4\dot{\boldsymbol{v}}^4\boldsymbol{n}_d-\frac{1}{72}r_d^4\ddot{\boldsymbol{v}}^2\boldsymbol{n}_d+\frac{1}{24}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$+\frac{3}{128}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4\boldsymbol{n}_d-\frac{1}{144}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{48}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{n}_d$$

$$
\begin{aligned}
&- \frac{1}{120} r_d^4 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{24} r_d^4 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{1}{48} r_d^4 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{1}{64} r_d^4 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d - \frac{1}{24} r_d^4 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{24} r_d^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{48} r_d^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{24} r_d^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{12} r_d^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} + \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \\
&+ \frac{1}{48} r_d^3 r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{5}{32} r_d^3 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{1}{24} r_d^3 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{48} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{3}{32} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{24} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{32} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{8} r_d^3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{5}{32} r_d^3 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{1}{8} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} \\
&- \frac{3}{32} r_d^2 r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{3}{32} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&- \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{16} r_d r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} \\
&- \frac{1}{16} r_d r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d + \mathrm{O}(\varepsilon^5). \qquad \text{(G.42)}
\end{aligned}
$$

G.6.11 Retarded velocity and derivatives

Taking successive t-derivatives of (G.30), we find

$$
\begin{aligned}
\boldsymbol{v}_{r'}(t) &= t\dot{\boldsymbol{v}} + \frac{1}{2} t^2 \ddot{\boldsymbol{v}} - t(\boldsymbol{r}' \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{1}{6} t^3 \dddot{\boldsymbol{v}} - t^2 (\boldsymbol{r}' \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} - \frac{1}{2} t^2 (\boldsymbol{r}' \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ t(\boldsymbol{r}' \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} + \frac{1}{24} t^4 \ddddot{\boldsymbol{v}} - \frac{1}{2} t^3 (\boldsymbol{r}' \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{v}} - \frac{1}{2} t^3 (\boldsymbol{r}' \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} - \frac{1}{6} t^3 (\boldsymbol{r}' \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&- \frac{1}{2} t^3 \dot{\boldsymbol{v}}^2 (\boldsymbol{r}' \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{3}{2} t^2 (\boldsymbol{r}' \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} + \frac{3}{2} t^2 (\boldsymbol{r}' \cdot \dot{\boldsymbol{v}}) (\boldsymbol{r}' \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - t(\boldsymbol{r}' \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}}
\end{aligned}
$$

$$+ \mathrm{O}(\varepsilon^5), \tag{G.43}$$

$$\begin{aligned}
\dot{\boldsymbol{v}}_{r'}(t) \;&=\; \dot{\boldsymbol{v}} + t\ddot{\boldsymbol{v}} - (\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2\dddot{\boldsymbol{v}} - 2t(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - t(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + (\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&\quad + \frac{1}{6}t^3\ddddot{\boldsymbol{v}} - \frac{3}{2}t^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - \frac{3}{2}t^2(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{2}t^2(\boldsymbol{r}'\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{3}{2}t^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&\quad + 3t(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 3t(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - (\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^4),
\end{aligned} \tag{G.44}$$

$$\begin{aligned}
\ddot{\boldsymbol{v}}_{r'}(t) \;&=\; \ddot{\boldsymbol{v}} + t\dddot{\boldsymbol{v}} - 2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - (\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}t^2\ddddot{\boldsymbol{v}} - 3t(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} - 3t(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&\quad - t(\boldsymbol{r}'\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 3t\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} + 3(\boldsymbol{r}'\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&\quad + \mathrm{O}(\varepsilon^3).
\end{aligned} \tag{G.45}$$

Substituting $t = t_{\mathrm{ret}}$ from (G.32), by using (G.38) in (G.43), (G.44) and (G.45), we find

$$\begin{aligned}
\boldsymbol{v}_{\mathrm{ret}} \;&=\; -r_d\dot{\boldsymbol{v}} + \frac{1}{2}r_d^2\ddot{\boldsymbol{v}} + \frac{1}{2}r_dr_s(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{6}r_d^3\dddot{\boldsymbol{v}} - \frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{1}{8}r_d^3(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&\quad + \frac{1}{12}r_d^3(\boldsymbol{n}_d\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^2r_s(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{4}r_d^2r_s(\boldsymbol{n}_s\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{4}r_dr_s^2(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&\quad + \frac{1}{24}r_d^4\ddddot{\boldsymbol{v}} + \frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + \frac{1}{8}r_d^4(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} - \frac{1}{12}r_d^4(\boldsymbol{n}_d\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&\quad - \frac{1}{24}r_d^4(\boldsymbol{n}_d\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{12}r_d^4(\dot{\boldsymbol{v}}\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&\quad + \frac{1}{8}r_d^4(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}r_d^3r_s(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}} + \frac{1}{4}r_d^3r_s(\boldsymbol{n}_s\!\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&\quad + \frac{1}{12}r_d^3r_s(\boldsymbol{n}_s\!\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{7}{16}r_d^3r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{24}r_d^3r_s(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&\quad + \frac{3}{16}r_d^3r_s(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}r_d^3r_s(\boldsymbol{n}_d\!\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}} \\
&\quad + \frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}r_dr_s^3(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^5),
\end{aligned} \tag{G.46}$$

$$\dot{\boldsymbol{v}}_{\mathrm{ret}} \;=\; \dot{\boldsymbol{v}} - r_d\ddot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_s(\boldsymbol{n}_s\!\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_d^2\dddot{\boldsymbol{v}} - \frac{1}{2}r_d^2(\boldsymbol{n}_d\!\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}$$

$$
\begin{aligned}
&+\frac{1}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}-\frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{2}r_dr_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{1}{2}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{4}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}-\frac{1}{6}r_d^3\dddot{\boldsymbol{v}}-\frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}-\frac{3}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{1}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}+\frac{7}{12}r_d^3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d^3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_d^2r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}-\frac{3}{4}r_d^2r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{1}{4}r_d^2r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_d^2r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{2}r_d^2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{8}r_d^2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^2r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{3}{4}r_dr_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\
&+\frac{3}{8}r_dr_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}-\frac{3}{4}r_dr_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{8}r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}\\
&+\mathrm{O}(\varepsilon^4), \qquad \text{(G.47)}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\boldsymbol{v}}_{\text{ret}} = {}& \ddot{\boldsymbol{v}}-r_d\dddot{\boldsymbol{v}}+r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{1}{2}r_d^2\ddddot{\boldsymbol{v}}-r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}+\frac{3}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}-\frac{3}{2}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&-\frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{3}{2}r_d^2\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}+\frac{3}{2}r_dr_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{2}r_dr_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{2}r_dr_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{3}{4}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_dr_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{3}{4}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\mathrm{O}(\varepsilon^3). \qquad \text{(G.48)}
\end{aligned}
$$

G.6.12 Gamma factor

Computing $\gamma_{\text{ret}}^{-2}\equiv 1-\boldsymbol{v}_{\text{ret}}^2$ from (G.46), we find

$$
\gamma_{\text{ret}}^{-2} = 1-r_d^2\dot{\boldsymbol{v}}^2+r_d^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})+r_d^2r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})-\frac{1}{4}r_d^4\dot{\boldsymbol{v}}^4-\frac{1}{4}r_d^4\ddot{\boldsymbol{v}}^2-\frac{1}{3}r_d^4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})
$$

$$
\begin{aligned}
&- \frac{1}{4} r_d^4 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 + \frac{1}{6} r_d^4 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) - \frac{1}{2} r_d^3 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \\
&- \frac{3}{2} r_d^3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) - \frac{3}{4} r_d^2 r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 + \frac{1}{12} r_d^5 (\dot{\boldsymbol{v}} \cdot \ddddot{\boldsymbol{v}}) + \frac{1}{6} r_d^5 (\ddot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}}) \\
&- \frac{1}{12} r_d^5 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{v}}) + \frac{13}{24} r_d^5 \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) - \frac{1}{4} r_d^5 \dot{\boldsymbol{v}}^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \\
&+ \frac{3}{8} r_d^5 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) - \frac{1}{4} r_d^5 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) + \frac{1}{4} r_d^5 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \\
&+ \frac{1}{6} r_d^4 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) + r_d^4 r_s \dot{\boldsymbol{v}}^4 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) + \frac{1}{2} r_d^4 r_s \ddot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \\
&+ \frac{2}{3} r_d^4 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}}) + \frac{3}{4} r_d^4 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) - \frac{1}{12} r_d^4 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \\
&+ \frac{1}{2} r_d^4 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) - \frac{1}{3} r_d^4 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \\
&+ \frac{3}{2} r_d^3 r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) + r_d^3 r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) + \frac{1}{2} r_d^2 r_s^3 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \\
&+ \mathrm{O}(\varepsilon^6).
\end{aligned}
$$

G.6.13 Modified retarded normal vectors

We now compute $\boldsymbol{n}'$ and $\boldsymbol{n}''$ via

$$
\begin{aligned}
\boldsymbol{n}' &\equiv \boldsymbol{n} - \boldsymbol{v}_{\mathrm{ret}}, \\
\boldsymbol{n}'' &\equiv \boldsymbol{n}' - \boldsymbol{v}_{\mathrm{ret}} \times (\boldsymbol{n} \times \boldsymbol{v}_{\mathrm{ret}});
\end{aligned}
\qquad \text{(G.49)}
$$

using (G.42) and (G.46), we find

$$
\begin{aligned}
\boldsymbol{n}' = \;& \boldsymbol{n}_d + \frac{1}{2} r_d \dot{\boldsymbol{v}} + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{3} r_d^2 \ddot{\boldsymbol{v}} - \frac{1}{8} r_d^2 \dot{\boldsymbol{v}}^2 \boldsymbol{n}_d + \frac{1}{8} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{1}{6} r_d^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{4} r_d r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{1}{4} r_d r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{8} r_d^3 \dddot{\boldsymbol{v}} \\
&+ \frac{1}{16} r_d^3 \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}} + \frac{1}{16} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} + \frac{1}{16} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d - \frac{1}{12} r_d^3 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{24} r_d^3 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{12} r_d^3 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{16} r_d^3 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{12} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{3} r_d^2 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} + \frac{1}{6} r_d^2 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}r_d^2 r_s \dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{12}r_d^2 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{8}r_d^2 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{6}r_d^2 r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\\
&+\frac{1}{8}r_d r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{30}r_d^4\ddddot{\boldsymbol{v}}-\frac{1}{12}r_d^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}-\frac{5}{128}r_d^4\dot{\boldsymbol{v}}^4\boldsymbol{n}_d\\
&-\frac{1}{72}r_d^4\ddot{\boldsymbol{v}}^2\boldsymbol{n}_d-\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{3}{128}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4\boldsymbol{n}_d+\frac{11}{144}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&+\frac{1}{48}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{1}{24}r_d^4(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{120}r_d^4(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{24}r_d^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{48}r_d^4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{1}{64}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{24}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{1}{48}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{24}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{3}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}-\frac{3}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&-\frac{1}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{9}{32}r_d^3 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{24}r_d^3 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{1}{24}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{48}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{32}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{24}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{32}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d^3 r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{1}{16}r_d^3 r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{16}r_d^3 r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{8}r_d^3 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{5}{32}r_d^3 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{1}{8}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{4}r_d^2 r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\
&-\frac{3}{32}r_d^2 r_s^2\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{3}{32}r_d^2 r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\
&-\frac{1}{8}r_d^2 r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{4}r_d^2 r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{1}{8}r_d^2 r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{16}r_d r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
-\frac{1}{16}r_d r_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d+\mathrm{O}(\varepsilon^5), \tag{G.50}
$$

$$
\begin{aligned}
\boldsymbol{n}'' = {} & \boldsymbol{n}_d+\frac{1}{2}r_d\dot{\boldsymbol{v}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{3}r_d^2\ddot{\boldsymbol{v}}-\frac{9}{8}r_d^2\dot{\boldsymbol{v}}^2\boldsymbol{n}_d+r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& +\frac{1}{8}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{6}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{4}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& -\frac{1}{4}r_d r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d^3\dddot{\boldsymbol{v}}+\frac{1}{16}r_d^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}-\frac{1}{2}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& +\frac{9}{16}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}+\frac{1}{16}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d-\frac{7}{12}r_d^3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& +\frac{1}{24}r_d^3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{13}{12}r_d^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{7}{16}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
& -\frac{1}{12}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{3}r_d^2 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{6}r_d^2 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& +\frac{9}{8}r_d^2 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-r_d^2 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{12}r_d^2 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& -\frac{1}{8}r_d^2 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{6}r_d^2 r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
& +\frac{1}{8}r_d r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{30}r_d^4\ddddot{\boldsymbol{v}}-\frac{21}{128}r_d^4\dot{\boldsymbol{v}}^4\boldsymbol{n}_d-\frac{19}{72}r_d^4\ddot{\boldsymbol{v}}^2\boldsymbol{n}_d \\
& +\frac{1}{6}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}-\frac{1}{3}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{3}{8}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}+\frac{3}{128}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4\boldsymbol{n}_d \\
& +\frac{47}{144}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{48}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{5}{24}r_d^4(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& -\frac{1}{120}r_d^4(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{8}r_d^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{17}{48}r_d^4(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{4}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& -\frac{23}{64}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{7}{24}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{2}{3}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
& +\frac{1}{48}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{11}{24}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& -\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{3}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}-\frac{3}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
& -\frac{1}{16}r_d^3 r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{9}{32}r_d^3 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{13}{24}r_d^3 r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& +\frac{3}{4}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{13}{24}r_d^3 r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{48} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{27}{32} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{24} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{32} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{7}{8} r_d^3 r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{16} r_d^3 r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{13}{8} r_d^3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{19}{32} r_d^3 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{8} r_d^3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{4} r_d^2 r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} - \frac{27}{32} r_d^2 r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{3}{4} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&+ \frac{3}{32} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d - \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{1}{4} r_d^2 r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{1}{8} r_d^2 r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{16} r_d r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} - \frac{1}{16} r_d r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d + \mathrm{O}(\varepsilon^5). \qquad \text{(G.51)}
\end{aligned}
$$

G.6.14 Retarded Doppler factor

We now compute the retarded "Doppler factor" κ via

$$
\kappa \equiv \frac{1}{1 - (\boldsymbol{n} \cdot \boldsymbol{v}_{\text{ret}})};
$$

from (G.42) and (G.46), we find

$$
\begin{aligned}
\kappa \;=\; & 1 - r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) + \frac{1}{2} r_d^2 \dot{\boldsymbol{v}}^2 + \frac{1}{2} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 + \frac{1}{2} r_d^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \\
&+ \frac{1}{2} r_d r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) - \frac{1}{4} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 - \frac{1}{6} r_d^3 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) - \frac{5}{12} r_d^3 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \\
&- r_d^3 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) - \frac{1}{2} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) - \frac{1}{2} r_d^2 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \\
&- \frac{1}{4} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) - \frac{1}{2} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) - \frac{1}{2} r_d^2 r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \\
&- \frac{1}{4} r_d r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 + \frac{3}{8} r_d^4 \dot{\boldsymbol{v}}^4 + \frac{1}{12} r_d^4 \ddot{\boldsymbol{v}}^2 + \frac{1}{8} r_d^4 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^4 \\
&+ \frac{1}{12} r_d^4 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})^2 + \frac{1}{24} r_d^4 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) + \frac{1}{8} r_d^4 (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}}) + \frac{7}{8} r_d^4 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{25}{48}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})+\frac{1}{6}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})+\frac{5}{6}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\\
&+\frac{7}{16}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})+\frac{5}{24}r_d^3r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})+\frac{1}{12}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\\
&+\frac{1}{4}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})+\frac{3}{8}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})+\frac{1}{4}r_d^3r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\\
&+\frac{1}{4}r_d^3r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})+\frac{5}{8}r_d^3r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})+\frac{7}{4}r_d^3r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\\
&+\frac{3}{4}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})+\frac{3}{8}r_d^2r_s^2\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\\
&+\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2+\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\\
&+\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})+\frac{1}{8}r_dr_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3+\mathrm{O}(\varepsilon^5). \qquad (\text{G.52})
\end{aligned}
$$

G.6.15 Final redshift expression

Finally, we write the redshift factor λ of (G.27) in terms of r_d, $\boldsymbol{n}_d$, r_s and $\boldsymbol{n}_s$:

$$
\lambda = 1+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})+\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}). \qquad (\text{G.53})
$$

G.6.16 Spin of each constituent

The three-spin of the centre of the body, $\boldsymbol{\sigma}$, has, by definition, the parametrisation

$$
\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}+t\dot{\boldsymbol{\sigma}}+\frac{1}{2}t^2\ddot{\boldsymbol{\sigma}}+\frac{1}{6}t^3\dddot{\boldsymbol{\sigma}}+\frac{1}{24}t^4\ddddot{\boldsymbol{\sigma}}+\mathrm{O}(\varepsilon^5). \qquad (\text{G.54})
$$

Using $t(\tau)$ from (G.14), we find

$$
\begin{aligned}
\boldsymbol{\sigma}(\tau) = {}& \boldsymbol{\sigma}+\tau\dot{\boldsymbol{\sigma}}+\frac{1}{2}\tau^2\ddot{\boldsymbol{\sigma}}+\frac{1}{6}\tau^3\dddot{\boldsymbol{\sigma}}+\frac{1}{6}\tau^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}+\frac{1}{24}\tau^4\ddddot{\boldsymbol{\sigma}}+\frac{1}{6}\tau^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{8}\tau^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\mathrm{O}(\varepsilon^5). \qquad (\text{G.55})
\end{aligned}
$$

Now, the three-spin $\boldsymbol{\sigma}_r(\tau)$ for any constituent $\boldsymbol{r}$ at body proper-time τ is equal to that of the body as a whole, *viz.*,

$$
\boldsymbol{\sigma}_r(\tau) \equiv \boldsymbol{\sigma}(\tau). \qquad (\text{G.56})
$$

Using (G.28) and (G.55) in (G.56), we find

$$
\begin{aligned}
\boldsymbol{\sigma}_{r'}(t) \;=\; & \boldsymbol{\sigma} + t\dot{\boldsymbol{\sigma}} + \frac{1}{2}t^2\ddot{\boldsymbol{\sigma}} - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{1}{6}t^3\dddot{\boldsymbol{\sigma}} - t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
& + t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{24}t^4\ddddot{\boldsymbol{\sigma}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{6}t^3(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
& - \frac{1}{2}t^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}} \\
& + \mathrm{O}(\varepsilon^5); \qquad \text{(G.57)}
\end{aligned}
$$

G.6.17 Constituent retarded FitzGerald spins

We can compute the retarded FitzGerald spin vector as a function of t, via (G.6); using (G.46) and (G.57), we find

$$
\begin{aligned}
\boldsymbol{\sigma}'_{r'}(t) \;=\; & \boldsymbol{\sigma} + t\dot{\boldsymbol{\sigma}} + \frac{1}{2}t^2\ddot{\boldsymbol{\sigma}} - \frac{1}{2}t^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{1}{6}t^3\dddot{\boldsymbol{\sigma}} - \frac{1}{4}t^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
& - \frac{1}{2}t^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{4}t^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
& + t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{24}t^4\ddddot{\boldsymbol{\sigma}} - \frac{1}{12}t^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} \\
& - \frac{1}{4}t^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} - \frac{1}{4}t^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{8}t^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{1}{4}t^4(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
& - \frac{1}{12}t^4(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{8}t^4\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{1}{2}t^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} \\
& - \frac{1}{6}t^3(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{1}{2}t^3\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{4}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
& + \frac{3}{2}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{3}{4}t^3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}t^3(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& - t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}} + \mathrm{O}(\varepsilon^5). \qquad \text{(G.58)}
\end{aligned}
$$

Taking successive lab-time derivatives of (G.58), we find

$$
\dot{\boldsymbol{\sigma}}'_{r'}(t) \;=\; \dot{\boldsymbol{\sigma}} + t\ddot{\boldsymbol{\sigma}} - t(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{1}{2}t^2\dddot{\boldsymbol{\sigma}} - \frac{3}{4}t^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}
$$

$$
\begin{aligned}
&- \frac{3}{2}t^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{3}{4}t^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 2t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - t(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ 2t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{6}t^3\dddot{\boldsymbol{\sigma}} - \frac{1}{3}t^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} - t^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} \\
&- t^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{2}t^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - t^3(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{3}t^3(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{1}{2}t^3\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{3}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2}t^2(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- \frac{3}{2}t^2\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{9}{4}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{9}{2}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{9}{4}t^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}t^2(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}} \\
&+ 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - (\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}} + \mathrm{O}(\varepsilon^4), \qquad \text{(G.59)}
\end{aligned}
$$

$$
\begin{aligned}
\ddot{\boldsymbol{\sigma}}'_{r'}(t) \;=\;& \ddot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + t\dddot{\boldsymbol{\sigma}} - \frac{3}{2}t(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3t(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{3}{2}t(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - (\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + 2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}t^2\ddddot{\boldsymbol{\sigma}} - t^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} \\
&- 3t^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} - 3t^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{3}{2}t^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3t^2(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- t^2(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{2}t^2\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 3t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - 3t(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - t(\boldsymbol{r}'\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- 3t\dot{\boldsymbol{v}}^2(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{9}{2}t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + 9t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{9}{2}t(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3t(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}} + 3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})(\boldsymbol{r}'\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- 3(\boldsymbol{r}'\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^3). \qquad \text{(G.60)}
\end{aligned}
$$

Substituting $t = t_{\text{ret}} \equiv -R$, we find

$$
\begin{aligned}
\boldsymbol{\sigma}'_{\text{ret}} \;=\;& \boldsymbol{\sigma} - r_d\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^2\ddot{\boldsymbol{\sigma}} - \frac{1}{2}r_d^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}r_d r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{1}{6}r_d^3\dddot{\boldsymbol{\sigma}} \\
&- \frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} - \frac{1}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{12}r_d^3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{1}{4}r_d^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{1}{4}r_d^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^2 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{4}r_d^2 r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2}r_d^2 r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{4}r_d r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{24}r_d^4\ddddot{\boldsymbol{\sigma}} + \frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}-\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{1}{24}r_d^4(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{12}r_d^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{12}r_d^4(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-\frac{1}{4}r_d^4(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-\frac{1}{4}r_d^4(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{1}{8}r_d^4(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-\frac{1}{4}r_d^4(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{1}{12}r_d^4(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{1}{8}r_d^4\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{4}r_d^4\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{1}{8}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{8}r_d^4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{1}{12}r_d^4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d^3r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}}+\frac{1}{4}r_d^3r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{12}r_d^3r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{7}{16}r_d^3r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{24}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{16}r_d^3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{8}r_d^3r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{3}{8}r_d^3r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{3}{4}r_d^3r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{3}{8}r_d^3r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{1}{4}r_d^3r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}+\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{3}{8}r_d^2r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{1}{8}r_dr_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}}+\mathrm{O}(\varepsilon^5),\qquad\text{(G.61)}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\boldsymbol{\sigma}}'_{\mathrm{ret}}=\;&\dot{\boldsymbol{\sigma}}-r_d\ddot{\boldsymbol{\sigma}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^2\dddot{\boldsymbol{\sigma}}\\
&-\frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{4}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{4}r_d^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-\frac{3}{2}r_d^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{3}{4}r_d^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{1}{2}r_d^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}r_dr_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_dr_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-r_dr_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{1}{4}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}-\frac{1}{6}r_d^3\ddddot{\boldsymbol{\sigma}}-\frac{1}{8}r_d^3\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}}+\frac{1}{4}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}}\\
&-\frac{3}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}+\frac{1}{8}r_d^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}}+\frac{7}{12}r_d^3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{4}r_d^3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{3}r_d^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}+r_d^3(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}+r_d^3(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{1}{2}r_d^3(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+r_d^3(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+\frac{1}{3}r_d^3(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{5}{8}r_d^3\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{3}{8} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{3}{4} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- \frac{3}{8} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{8} r_d^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{7}{12} r_d^3 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{4} r_d^2 r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_d^2 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{2} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- \frac{3}{8} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{9}{8} r_d^2 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{9}{4} r_d^2 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{9}{8} r_d^2 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{4} r_d^2 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} + \frac{3}{8} r_d r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{\sigma}} - \frac{3}{4} r_d r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{3}{4} r_d r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{8} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{\sigma}} + \mathrm{O}(\varepsilon^4),
\end{aligned}
\tag{G.62}
$$

$$
\begin{aligned}
\ddot{\boldsymbol{\sigma}}'_{\text{ret}} = {} & \ddot{\boldsymbol{\sigma}} - (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - r_d \dddot{\boldsymbol{\sigma}} + r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{2} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ 3 r_d (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{3}{2} r_d (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} \\
&- \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2} r_d^2 \ddddot{\boldsymbol{\sigma}} - r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} \\
&+ \frac{3}{4} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} - \frac{3}{2} r_d^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2} r_d^2 (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - r_d^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dddot{\boldsymbol{v}} \\
&- 3 r_d^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} - 3 r_d^2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{3}{2} r_d^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3 r_d^2 (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- r_d^2 (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{2} r_d^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{3}{2} r_d^2 \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{2} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} + 3 r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{3}{2} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{4} r_d^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2} r_d^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{3}{2} r_d r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} + \frac{3}{2} r_d r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{2} r_d r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{2} r_d r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{3}{2} r_d r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} - \frac{3}{4} r_d r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_d r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{9}{4} r_d r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{9}{2} r_d r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_d r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{3}{2} r_d r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&+ \frac{3}{2} r_d r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{3}{4} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} + \frac{3}{4} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^3). \qquad \text{(G.63)}
\end{aligned}
$$

G.6.18 General retarded field expressions

The retarded field expressions verified explicitly in the program RETFIELD are

$$
\boldsymbol{E}_1'^q = -\kappa^3 (\boldsymbol{n} \cdot \boldsymbol{n}') \dot{\boldsymbol{v}} + \kappa^3 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}',
$$

$$
\boldsymbol{E}_2'^q = \kappa^3 \gamma^{-2} \boldsymbol{n}',
$$

$$
\begin{aligned}
\boldsymbol{E}_1'^d = {}& 3\kappa^5 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n} \cdot \boldsymbol{\sigma}') \boldsymbol{n}' - 3\kappa^5 (\boldsymbol{n} \cdot \boldsymbol{n}')(\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\boldsymbol{n} \cdot \boldsymbol{\sigma}') \dot{\boldsymbol{v}} - \kappa^4 (\boldsymbol{n} \cdot \boldsymbol{n}')(\boldsymbol{n} \cdot \boldsymbol{\sigma}') \ddot{\boldsymbol{v}} \\
&- 3\kappa^4 (\boldsymbol{n} \cdot \boldsymbol{n}')(\boldsymbol{n} \cdot \dot{\boldsymbol{\sigma}}') \dot{\boldsymbol{v}} + 3\kappa^4 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\boldsymbol{n} \cdot \dot{\boldsymbol{\sigma}}') \boldsymbol{n}' + \kappa^4 (\boldsymbol{n} \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n} \cdot \boldsymbol{\sigma}') \boldsymbol{n}' \\
&- \kappa^3 (\boldsymbol{n} \cdot \boldsymbol{n}') \ddot{\boldsymbol{\sigma}}' - \kappa^3 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}}' + \kappa^3 (\boldsymbol{n} \cdot \dot{\boldsymbol{\sigma}}') \dot{\boldsymbol{v}} + \kappa^3 (\boldsymbol{n} \cdot \ddot{\boldsymbol{\sigma}}') \boldsymbol{n}',
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_2'^d = {}& -3\kappa^5 (\boldsymbol{n} \cdot \boldsymbol{n}')(\boldsymbol{n}'' \cdot \boldsymbol{\sigma}') \dot{\boldsymbol{v}} + 3\kappa^5 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}'' \cdot \boldsymbol{\sigma}') \boldsymbol{n}' + 3\kappa^5 (\boldsymbol{n} \cdot \boldsymbol{\sigma}')(\boldsymbol{n}'' \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}' \\
&+ 3\kappa^4 (\boldsymbol{n}'' \cdot \dot{\boldsymbol{\sigma}}') \boldsymbol{n}' - \kappa^3 \gamma^{-2} \dot{\boldsymbol{\sigma}}' - \kappa^3 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma}' + \kappa^3 (\boldsymbol{n}' \cdot \boldsymbol{\sigma}') \dot{\boldsymbol{v}} \\
&+ \kappa^3 (\boldsymbol{v} \cdot \dot{\boldsymbol{\sigma}}') \boldsymbol{n}' - \kappa^3 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}') \boldsymbol{n}',
\end{aligned}
$$

$$
\boldsymbol{E}_3'^d = 3\kappa^5 \gamma^{-2} (\boldsymbol{n}'' \cdot \boldsymbol{\sigma}') \boldsymbol{n}' - \kappa^3 \gamma^{-2} \boldsymbol{\sigma}',
$$

$$
\boldsymbol{B}_1'^q = -\kappa^3 (\boldsymbol{n} \cdot \boldsymbol{n}') \boldsymbol{n} \times \dot{\boldsymbol{v}} + \kappa^3 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{n} \times \boldsymbol{n}',
$$

$$\boldsymbol{B}_2'^q = \kappa^3\gamma^{-2}\boldsymbol{n}\times\boldsymbol{n}',$$

$$\begin{aligned}\boldsymbol{B}_1'^d &= 3\kappa^5(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\boldsymbol{n}' - 3\kappa^5(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\dot{\boldsymbol{v}}\\ &\quad - \kappa^4(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\ddot{\boldsymbol{v}} - 3\kappa^4(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\dot{\boldsymbol{v}}\\ &\quad + 3\kappa^4(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\boldsymbol{n}' + \kappa^4(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\boldsymbol{n}' - \kappa^3(\boldsymbol{n}\cdot\boldsymbol{n}')\boldsymbol{n}\times\ddot{\boldsymbol{\sigma}}'\\ &\quad - \kappa^3(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}' + \kappa^3(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\dot{\boldsymbol{v}} + \kappa^3(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\boldsymbol{n}',\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}_2'^d &= -3\kappa^5(\boldsymbol{n}\cdot\boldsymbol{n}')(\boldsymbol{n}''\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\dot{\boldsymbol{v}} + 3\kappa^5(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}''\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\boldsymbol{n}'\\ &\quad + 3\kappa^5(\boldsymbol{n}\cdot\boldsymbol{\sigma}')(\boldsymbol{n}''\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\boldsymbol{n}' + 3\kappa^4(\boldsymbol{n}''\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\boldsymbol{n}' - \kappa^3\gamma^{-2}\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}'\\ &\quad - \kappa^3(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\boldsymbol{\sigma}' + \kappa^3(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}'\times\boldsymbol{\sigma}' - \kappa^3(\boldsymbol{n}\cdot\boldsymbol{\sigma}')\boldsymbol{n}'\times\dot{\boldsymbol{v}}\\ &\quad + \kappa^3(\boldsymbol{n}'\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\dot{\boldsymbol{v}} + \kappa^3(\boldsymbol{v}\cdot\dot{\boldsymbol{\sigma}}')\boldsymbol{n}\times\boldsymbol{n}' - \kappa^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}')\boldsymbol{n}\times\boldsymbol{n}' - \kappa^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}',\end{aligned}$$

$$\boldsymbol{B}_3'^d = -3\kappa^5\gamma^{-2}(\boldsymbol{n}''\cdot\boldsymbol{\sigma}')\boldsymbol{n}'\times\boldsymbol{v} - \kappa^3\gamma^{-2}\boldsymbol{v}\times\boldsymbol{\sigma}',$$

G.6.19 The retarded self-fields

Using the expressions in the previous sections, we find

$$\begin{aligned}\boldsymbol{E}_1^q &= -r_d^{-1}\dot{\boldsymbol{v}} + r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \ddot{\boldsymbol{v}} - \frac{1}{2}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\ &\quad - (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2}r_d\dddot{\boldsymbol{v}}\\ &\quad - \frac{9}{8}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{4}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{11}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{5}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d\\ &\quad - \frac{5}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{15}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\ &\quad + \frac{3}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\ &\quad - \frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\ &\quad + r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} + \frac{1}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\ &\quad + \mathrm{O}(\varepsilon^2),\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{E}_2^q \;=\;& r_d^{-2}\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}} - \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{3}\ddot{\boldsymbol{v}} + \frac{1}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d - (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{9}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{8}r_d\dddot{\boldsymbol{v}} + \frac{3}{16}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + \frac{2}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&+ \frac{17}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{11}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{3}{8}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{21}{16}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{3}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&+ \frac{1}{6}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{1}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{9}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{1}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{3}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_1^d \;=\;& -r_d^{-1}\ddot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \dddot{\boldsymbol{\sigma}} + \frac{1}{2}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} \\
&+ \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} \\
&+ \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - (\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{1}{2}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{13}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{7}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{5}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{7}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&+(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{13}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+2r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d\dddot{\boldsymbol{\sigma}}-\frac{11}{8}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}-r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{11}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-\frac{17}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}\\
&+\frac{3}{2}r_d(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}+\frac{7}{4}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{1}{2}r_d(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{4}{3}r_d(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&+\frac{8}{3}r_d(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{1}{6}r_d(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+r_d(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{1}{2}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{11}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{19}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{21}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{27}{8}r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{3}{4}r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{3}{4}r_d\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{3}{4}r_d\dot{\boldsymbol{v}}^4(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{6}r_d\ddot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{7}{6}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-7r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-\frac{11}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{25}{12}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+\frac{15}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}
\end{aligned}
$$

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\begin{aligned}
&- r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{25}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{39}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{39}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{25}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{9}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{57}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{27}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{15}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{33}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- 3r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{55}{12}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + 3r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{13}{12}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{5}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{4}{3}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{17}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{3}{2}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ 2r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{9}{4}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{63}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{33}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{37}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{81}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{17}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{37}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{21}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{7}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{19}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{23}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{3}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{3}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{9}{4}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- \frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}
\end{aligned}
$$

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\begin{aligned}
&- \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{v}} - \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \\
&- \frac{1}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{9}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} - \frac{11}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{15}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{3}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} + 3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{9}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{5}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{15}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{1}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{15}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{3}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - 6 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \\
&- \frac{7}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{1}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{39}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{7}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{21}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{15}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{9}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{5}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - 9 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{13}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + 9 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- 3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{27}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{9}{2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{21}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ 6 r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d
\end{aligned}
$$

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\begin{aligned}
&+ \frac{39}{4} r_s (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \dot{\boldsymbol{\sigma}} - \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} \\
&- \frac{3}{2} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} + \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d - \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_2^d \;=\; & -r_d^{-2} \dot{\boldsymbol{\sigma}} - r_d^{-2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \boldsymbol{\sigma} - 2 r_d^{-2} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \dot{\boldsymbol{v}} + 3 r_d^{-2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
& - r_d^{-2} (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d + 6 r_d^{-2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d + r_d^{-1} \ddot{\boldsymbol{\sigma}} + \frac{1}{2} r_d^{-1} \dot{\boldsymbol{v}}^2 \boldsymbol{\sigma} \\
& + \frac{5}{2} r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} + r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} + r_d^{-1} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}}) \boldsymbol{\sigma} + 2 r_d^{-1} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \ddot{\boldsymbol{v}} \\
& + \frac{7}{2} r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - 3 r_d^{-1} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{5}{2} r_d^{-1} (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{3}{2} r_d^{-1} (\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
& + r_d^{-1} (\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d + 8 r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{21}{2} r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
& + 4 r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d - 12 r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d \\
& - 6 r_d^{-1} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
& + r_d^{-2} r_s (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{3}{2} r_d^{-2} r_s (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d - 3 r_d^{-2} r_s (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{2} \dddot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}-2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{9}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}-\frac{5}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{\sigma}-\frac{5}{3}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}-(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-\frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{2}{3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{25}{12}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+\frac{13}{4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{23}{12}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{1}{2}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{15}{8}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{3}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{13}{8}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-7(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-11(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{15}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{21}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{53}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{63}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{41}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{51}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{11}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{17}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{11}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{33}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+19(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{5}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-2r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{7}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{5}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-8r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{21}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-4r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+12r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+6r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}-\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\\
&+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{3}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{1}{6}r_d\ddddot{\boldsymbol{\sigma}}+\frac{5}{8}r_d\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}}+\frac{11}{16}r_d\dot{\boldsymbol{v}}^4\boldsymbol{\sigma}\\
&+\frac{1}{6}r_d\ddot{\boldsymbol{v}}^2\boldsymbol{\sigma}+\frac{11}{12}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}}+\frac{15}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}+\frac{13}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4\boldsymbol{\sigma}+\frac{13}{12}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{5}{12}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{\sigma}+\frac{2}{3}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddddot{\boldsymbol{v}}+\frac{11}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dddot{\boldsymbol{v}}+2r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}\\
&+\frac{13}{12}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{4}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{7}{24}r_d(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{23}{24}r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-\frac{5}{2}r_d(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-2r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{5}{12}r_d(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{3}{2}r_d(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{29}{12}r_d(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+r_d(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{5}{6}r_d(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{17}{24}r_d(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{1}{6}r_d(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{23}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{35}{16}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}+\frac{19}{12}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}+r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}\\
&+\frac{17}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{15}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{59}{16}r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{23}{8}r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{11}{8}r_d\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{3}{2}r_d\dot{\boldsymbol{v}}^4(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{1}{2}r_d\ddot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{8}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{5}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}+9r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}+7r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&-\frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{31}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{51}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{5}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{9}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{9}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{7}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ 11 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{135}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- \frac{81}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{103}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{45}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{43}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 15 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{129}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{9}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d - \frac{21}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^4 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{13}{3} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{79}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- \frac{21}{4} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{65}{24} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{11}{4} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{29}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{13}{2} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{9}{4} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{15}{4} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{9}{8} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d - r_d (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{13}{8} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{39}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{27}{2} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{67}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d - \frac{93}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{31}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + 17 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 23 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{49}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{33}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d - \frac{29}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- 29 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} + \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{15}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{3}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+3r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{11}{6}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{27}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{15}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{5}{2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}+\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&+\frac{1}{2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{11}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{15}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{25}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-\frac{39}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{23}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{9}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-2r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{5}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{57}{16}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{21}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{1}{4}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{9}{2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{51}{16}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{21}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+\frac{25}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{33}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-8r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{45}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{63}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{27}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{45}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{13}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d
\end{aligned}
$$

$$+\frac{159}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{19}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$-\frac{189}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{123}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$-\frac{153}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{33}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$-\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{51}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{33}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{135}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$-\frac{57}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}$$
$$+\frac{3}{8}r_d^{-1}r_s^2\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}+\frac{15}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}$$
$$+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}$$
$$+\frac{3}{2}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{21}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}$$
$$-\frac{9}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}$$
$$-\frac{15}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{9}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+6r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}$$
$$-\frac{63}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}+3r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$-9r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$-\frac{9}{2}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{3}{2}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$-\frac{9}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$-\frac{9}{2}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d^{-2}r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{\sigma}}$$

$$
\begin{aligned}
&+\frac{1}{8}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{\sigma}+\frac{1}{4}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}\\
&-\frac{3}{8}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d+\frac{1}{8}r_d^{-2}r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{4}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d+\mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{E}_3^d \;=\;& -r_d^{-3}\boldsymbol{\sigma}+3r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+r_d^{-2}\dot{\boldsymbol{\sigma}}+\frac{3}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{15}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-1}\ddot{\boldsymbol{\sigma}}-\frac{1}{8}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}-\frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{9}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}\\
&-r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{5}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{9}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{15}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{75}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+5r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{3}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{4}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{15}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{6}\dddot{\boldsymbol{\sigma}}\\
&+\frac{1}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}+\frac{3}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{5}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}+\frac{11}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{\sigma}+\frac{11}{12}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{8}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{8}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}+(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}+\frac{3}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{1}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{3}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{5}{4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{4}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{3}{4}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{3}{8}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{21}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{15}{16}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- \frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{21}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{9}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{111}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{39}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{51}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{135}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{19}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{7}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{15}{8}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{75}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{25}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{8}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{9}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{5}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{3}{8}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{9}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{5}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{15}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ 3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{75}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- 5r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} \\
&+ \frac{3}{8}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma} + \frac{3}{8}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{4} r_d^{-2} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{3}{8} r_d^{-2} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{15}{8} r_d^{-2} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d - \frac{1}{24} r_d \ddddot{\boldsymbol{\sigma}} - \frac{3}{16} r_d \dot{\boldsymbol{v}}^2 \ddot{\boldsymbol{\sigma}} \\
&+ \frac{17}{128} r_d \dot{\boldsymbol{v}}^4 \boldsymbol{\sigma} + \frac{1}{24} r_d \ddot{\boldsymbol{v}}^2 \boldsymbol{\sigma} - \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{\sigma}} - \frac{11}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} \\
&- \frac{7}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{\sigma}} - \frac{51}{128} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^4 \boldsymbol{\sigma} - \frac{5}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} - \frac{19}{48} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})^2 \boldsymbol{\sigma} \\
&- \frac{1}{3} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{1}{10} r_d (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{1}{10} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \ddddot{\boldsymbol{v}} - \frac{3}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dddot{\boldsymbol{v}} \\
&- \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} - \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} + \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{12} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{48} r_d (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{13}{48} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{v}} + \frac{3}{4} r_d (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} + \frac{5}{8} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&- \frac{1}{4} r_d (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{11}{24} r_d (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} + \frac{3}{4} r_d (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - \frac{1}{2} r_d (\ddot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&+ \frac{13}{48} r_d (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{3}{8} r_d (\dddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{10} r_d (\ddddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{11}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&- \frac{119}{64} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{11}{12} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{1}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} \\
&- \frac{3}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} + \frac{3}{16} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{31}{32} r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{9}{8} r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{5}{8} r_d \dot{\boldsymbol{v}}^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{183}{128} r_d \dot{\boldsymbol{v}}^4 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{11}{24} r_d \ddot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{9}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{9}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{v}} - 3 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} - \frac{9}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ \frac{5}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{13}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} \\
&+ \frac{21}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{7}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{7}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{19}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{57}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ \frac{81}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{103}{32} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{27}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{23}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{123}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{75}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{21}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{825}{128}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{77}{48}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-\frac{19}{8}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+\frac{9}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{31}{24}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{13}{8}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{59}{48}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{175}{48}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{7}{8}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{7}{4}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{5}{8}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{7}{16}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{5}{8}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{51}{16}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{21}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{33}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{819}{64}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{7}{2}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{31}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{49}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{29}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{75}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{5}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{35}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}}-\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{1}{12}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{5}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{9}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{5}{6}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{15}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{33}{32}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{11}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{9}{16}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{9}{16}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{9}{16}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{9}{16}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{3}{16}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&-\frac{7}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{1}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{9}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{9}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&+\frac{15}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{9}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{9}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{3}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{9}{16}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{7}{8}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{9}{16}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{3}{16}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{81}{32}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{27}{32}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{1}{2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{3}{4}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{63}{32}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}-\frac{11}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{15}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{27}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{17}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{45}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{63}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{27}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-2r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{333}{32}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{25}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{117}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{153}{32}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{405}{32}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{15}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{45}{16}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{57}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{21}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{45}{16}r_s(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d
\end{aligned}
$$

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\begin{aligned}
&- \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{315}{32} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{75}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} \\
&- \frac{3}{32} r_d^{-1} r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{\sigma}} \\
&- \frac{27}{32} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} - \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&+ \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d - \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&+ \frac{15}{16} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{9}{32} r_d^{-1} r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{27}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&+ \frac{45}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{225}{32} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&+ \frac{15}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{15}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{\sigma}} \\
&- \frac{3}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{\sigma} - \frac{3}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} \\
&+ \frac{3}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d - \frac{3}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{15}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_1^q = & -r_d^{-1}\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - \frac{1}{2}r_d\boldsymbol{n}_d\times\dddot{\boldsymbol{v}} - \frac{1}{3}r_d\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} - \frac{5}{4}r_d\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& - \frac{5}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{7}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& - \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - \frac{1}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_2^q = & \; r_d^{-1}\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{2}\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} - \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{1}{6}r_d\boldsymbol{n}_d\times\dddot{\boldsymbol{v}} + \frac{1}{12}r_d\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} + \frac{1}{4}r_d\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& + \frac{5}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{11}{12}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& + \frac{1}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{1}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_1^d = & -r_d^{-1}\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& - 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}} + \frac{1}{2}\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} + \frac{1}{2}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
& + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}} \\
& + \frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + 3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
& - 2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{7}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + 7(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - \frac{5}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{7}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+2r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{2}r_d\boldsymbol{n}_d\times\ddddot{\boldsymbol{\sigma}}-\frac{1}{2}r_d\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{\sigma}}\\
&-\frac{1}{6}r_d\ddot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}-\frac{5}{4}r_d\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}-\frac{1}{4}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}}\\
&-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}-\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}\\
&-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{3}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}\\
&-2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{7}{6}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{v}}-3r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-2r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{1}{3}r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}+\frac{3}{2}r_d(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{3}{2}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{4}{3}r_d(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{8}{3}r_d(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+r_d(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{1}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{21}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-\frac{21}{4}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{7}{2}r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}\\
&-\frac{3}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-6r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-6r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{7}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{7}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{39}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{39}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ 3 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{27}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- 3 r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - 6 r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{17}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - 2 r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{5}{2} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{33}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{21}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{3}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dddot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{\sigma}} - \frac{3}{2} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} - \frac{1}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} \\
&- \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} - \frac{9}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} - \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} - \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} - \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&- \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} - \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} \\
&- \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{1}{2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{9}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - 3 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{9}{2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{3}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ 3 r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{9}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{5}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{15}{4} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{21}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{7}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{21}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{15}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- 9 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{21}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} - \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} - \frac{3}{2} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_2^d = &- r_d^{-2} \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - r_d^{-2} \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - 3 r_d^{-2} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + r_d^{-1} \boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} \\
&+ \frac{3}{2} r_d^{-1} \dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + r_d^{-1} \ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + \frac{3}{2} r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&+ 3 r_d^{-1} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + 6 r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{5}{2} r_d^{-1} (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ \frac{21}{2} r_d^{-1} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{3}{2} r_d^{-2} r_s (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{2} \boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}} \\
&- \dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} - \frac{7}{6} \ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{2} \dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{8} \dot{\boldsymbol{v}}^2 \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - \frac{5}{4} \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&- (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} - \frac{9}{4} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{5}{8} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} \\
&- \frac{5}{4} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{2} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}}) \boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - \frac{7}{6} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&- \frac{3}{2} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dddot{\boldsymbol{v}} - \frac{1}{2} (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} - \frac{9}{2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
&- \frac{9}{2} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{7}{4} (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{7}{2} (\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{9}{4} (\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{3}{2} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{15}{2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
&- 15 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{11}{2} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{63}{4} (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{17}{2} (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma}) \boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- r_d^{-1} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} - \frac{3}{2} r_d^{-1} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - r_d^{-1} r_s (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} - \frac{1}{2} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&- r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} - \frac{3}{2} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&- 3 r_d^{-1} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{3}{2} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- 6 r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{5}{2} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{21}{2} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{1}{4} r_d^{-2} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&- \frac{1}{4} r_d^{-2} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{3}{4} r_d^{-2} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{1}{6} r_d \boldsymbol{n}_d \times \ddddot{\boldsymbol{\sigma}} + \frac{5}{12} r_d \dot{\boldsymbol{v}} \times \dddot{\boldsymbol{\sigma}} + \frac{2}{3} r_d \ddot{\boldsymbol{v}} \times \ddot{\boldsymbol{\sigma}} + \frac{13}{24} r_d \dddot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{1}{6} r_d \ddddot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{1}{4} r_d \dot{\boldsymbol{v}}^2 \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} + \frac{25}{16} r_d \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{9}{8} r_d \dot{\boldsymbol{v}}^2 \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dddot{\boldsymbol{\sigma}} \\
&+ \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{\sigma}} + \frac{5}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{5}{12} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} + \frac{33}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{7}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{3}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} + \frac{7}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{5}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} \\
&+ \frac{17}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{7}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddddot{\boldsymbol{v}} + \frac{7}{24} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \times \dddot{\boldsymbol{v}} \\
&+ 2 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} + \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} + 3 r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ 2 r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{4}{3} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{3}{4} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} \\
&- \frac{11}{24} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} - \frac{9}{4} r_d (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{9}{4} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{3}{2} r_d (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - 3 r_d (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{25}{24} r_d (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{1}{2} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} + \frac{19}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{15}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ \frac{15}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{29}{8} r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{7}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}+2r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}\\
&+\frac{39}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{39}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{13}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{13}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{21}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{81}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{81}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{51}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{129}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{21}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{21}{2}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{19}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{15}{4}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{27}{2}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+23r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}}+\frac{3}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}+\frac{7}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}+\frac{13}{12}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{15}{16}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{21}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}+\frac{27}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{13}{12}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{15}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{15}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{7}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}
\end{aligned}
$$

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\begin{aligned}
&+ \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} + \frac{9}{4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{27}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ \frac{9}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{27}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{21}{8} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{21}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{27}{8} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - 2 r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{9}{2} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{45}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ 8 r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{45}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{33}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{189}{8} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{51}{4} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \ddot{\boldsymbol{\sigma}} \\
&+ \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} + \frac{9}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} + \frac{9}{2} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} + \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&- \frac{15}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{63}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{9}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{1}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{3}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{B}_3^d = {} & r_d^{-2}\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - r_d^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{2}r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& - \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{15}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2}\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} \\
& + \frac{1}{2}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{6}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{3}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{5}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{11}{12}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}} \\
& + \frac{1}{4}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{3}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - \frac{3}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} - \frac{3}{2}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - (\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{15}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{39}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{19}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + 3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + \frac{15}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{6}r_d\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{\sigma}} - \frac{1}{4}r_d\ddot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} - \frac{1}{6}r_d\dddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
& - \frac{1}{24}r_d\ddddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{3}{8}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{3}{16}r_d\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} \\
& - \frac{3}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{11}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
& - \frac{11}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{7}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{5}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}
\end{aligned}
$$

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\begin{aligned}
&- \frac{5}{12} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{1}{3} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddddot{\boldsymbol{v}} \\
&- \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \times \dddot{\boldsymbol{v}} - \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} - \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{1}{12} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{1}{4} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} + \frac{1}{8} r_d (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} + \frac{3}{4} r_d (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{1}{2} r_d (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} + r_d (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{3}{8} r_d (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{11}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{3}{16} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&- \frac{3}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{9}{8} r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{5}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dddot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{v}} - \frac{15}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&- \frac{15}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{3}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ 3 r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{9}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{81}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{81}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{27}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{75}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{9}{4} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} - \frac{9}{2} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{13}{8} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{7}{4} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{21}{4} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} \\
&- \frac{49}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} - \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \ddot{\boldsymbol{\sigma}} \\
&- \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} - \frac{1}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} \\
&- \frac{1}{4} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{1}{12} r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{5}{8} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{9}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{5}{6}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{15}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{11}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}\\
&-\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-\frac{1}{4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-\frac{3}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{9}{4}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{9}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{9}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{7}{8}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{45}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-\frac{17}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{45}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{9}{2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{117}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{57}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{3}{4}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&-\frac{3}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{9}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{9}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{9}{4}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{3}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{9}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{45}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{9}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{8}r_d^{-2}r_s^3(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{3}{8}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon^2).
\end{aligned}
$$

If we add the various $\boldsymbol{E}^a_n$ and $\boldsymbol{B}^a_n$ for $n = 1, 2$ or $n = 1, 2, 3$ as appropriate, we find

$$
\begin{aligned}
4\pi\boldsymbol{E}^q \;=\;& r_d^{-2}\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{2}{3}\ddot{\boldsymbol{v}} - \frac{3}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d + \frac{1}{2}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{1}{8}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{8}r_d\dddot{\boldsymbol{v}} - \frac{15}{16}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{2}{3}r_d(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{5}{16}r_d(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&- \frac{1}{16}r_d(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^3\boldsymbol{n}_d - \frac{1}{3}r_d(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{8}r_d(\boldsymbol{n}_d\!\cdot\!\dddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{2}{3}r_d(\dot{\boldsymbol{v}}\!\cdot\!\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{9}{16}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{2}{3}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{1}{3}r_s(\boldsymbol{n}_s\!\cdot\!\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{3}{8}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{8}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{1}{8}r_d^{-1}r_s^2(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} - \frac{1}{8}r_d^{-1}r_s^2(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + \mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\boldsymbol{E}^d \;=\;& -r_d^{-3}\boldsymbol{\sigma} + 3r_d^{-3}(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-2}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-2}(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-2}(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{3}{2}r_d^{-2}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}\ddot{\boldsymbol{\sigma}} + \frac{3}{8}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} \\
&- \frac{1}{8}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2\boldsymbol{\sigma} - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{1}{4}r_d^{-1}(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{3}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ \frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{3}{4}r_d^{-2}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{2}{3}\dddot{\boldsymbol{\sigma}} + \frac{1}{4}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} \\
&+ \frac{1}{16}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^3\boldsymbol{\sigma} - \frac{1}{4}(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{1}{8}(\boldsymbol{n}_d\!\cdot\!\dddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{8}(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\dddot{\boldsymbol{v}} \\
&+ (\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} + \frac{3}{4}(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{2}{3}(\dot{\boldsymbol{v}}\!\cdot\!\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{1}{3}(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{3}{4}(\dot{\boldsymbol{v}}\!\cdot\!\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{1}{3}(\ddot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{2}(\ddot{\boldsymbol{v}}\!\cdot\!\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{1}{8}(\dddot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{9}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{15}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{3}{16}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{1}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{5}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{1}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{1}{8}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{9}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{8}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{1}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{8}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{8}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}\\
&-\frac{1}{8}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}+\frac{1}{8}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{8}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-\frac{3}{8}r_d\ddddot{\boldsymbol{\sigma}}-\frac{15}{16}r_d\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}}\\
&+\frac{105}{128}r_d\dot{\boldsymbol{v}}^4\boldsymbol{\sigma}+\frac{5}{24}r_d\ddot{\boldsymbol{v}}^2\boldsymbol{\sigma}-\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}}-\frac{3}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}}\\
&-\frac{3}{128}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^4\boldsymbol{\sigma}+\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{48}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{\sigma}+\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{15}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{4}{15}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddddot{\boldsymbol{v}}-r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dddot{\boldsymbol{v}}-\frac{4}{3}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}\\
&-\frac{2}{3}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{1}{8}r_d(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{5}{16}r_d(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{3}{16}r_d(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}\\
&-\frac{1}{4}r_d(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}+\frac{3}{8}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{7}{24}r_d(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&+r_d(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{2}{3}r_d(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{7}{16}r_d(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{1}{3}r_d(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{1}{15}r_d(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{21}{64}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}\\
&+\frac{2}{3}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-2r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}-3r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{15}{16} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{21}{32} r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + r_d \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&+ \frac{105}{128} r_d \dot{\boldsymbol{v}}^4 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{5}{24} r_d \ddot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{3}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{v}} - r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} - \frac{3}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{1}{6} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} + \frac{1}{2} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{3}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{3}{32} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&+ \frac{3}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{9}{128} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^4 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{13}{48} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{3}{8} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - \frac{1}{3} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{7}{48} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{1}{48} r_d (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{1}{8} r_d (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- 2 r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{5}{16} r_d (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{5}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{1}{3} r_d (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{21}{16} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{3}{8} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{21}{64} r_d \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{1}{4} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{16} r_d (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{\sigma}} - r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} - \frac{1}{3} r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{3} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{3}{8} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} + \frac{1}{16} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{3}{32} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{3}{8} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
&+ \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{9}{16} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dddot{\boldsymbol{v}} - \frac{9}{16} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{3}{2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{7}{8} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{9}{8} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ \frac{9}{8} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{1}{2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{3}{16} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{5}{8} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{3}{16} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{16} r_s (\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{33}{32} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{63}{32} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{15}{32} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d - \frac{1}{16} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{1}{4} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{8} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{15}{32} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{3}{32} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d + \frac{9}{32} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{8} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{16} r_s (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{33}{32} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{\sigma}} + \frac{9}{32} r_d^{-1} r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} \\
&- \frac{3}{32} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - \frac{3}{16} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{9}{32} r_d^{-1} r_s^2 \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&+ \frac{9}{32} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \\
&- \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{\sigma} \\
&+ \frac{1}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \dot{\boldsymbol{v}} - \frac{1}{16} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})\boldsymbol{n}_d
\end{aligned}
$$

$$
+\frac{3}{16}r_d^{-2}r_s^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d+\mathrm{O}(\varepsilon^2),
$$

$$
\begin{aligned}
4\pi\boldsymbol{B}^q \;=\; & \frac{1}{2}\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{1}{3}r_d\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}-\frac{1}{4}r_d\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-r_d\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
& -\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\boldsymbol{B}^d \;=\; & -r_d^{-2}\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
& +r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}}-\frac{2}{3}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
& -\frac{1}{3}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
& -\frac{1}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{1}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
& +\frac{1}{4}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-\frac{1}{4}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
& -\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
& -\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
& -\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
& -r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
& -\frac{1}{3}r_d\boldsymbol{n}_d\times\ddddot{\boldsymbol{\sigma}}-\frac{1}{4}r_d\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{\sigma}}+\frac{1}{4}r_d\ddot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}+\frac{3}{8}r_d\dddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{8}r_d\ddddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
& -r_d\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}+\frac{15}{16}r_d\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{15}{16}r_d\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
& +\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{3}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
& +\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}+\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
& +\frac{1}{8}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{6}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{8}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddddot{\boldsymbol{v}}\\
& -\frac{1}{3}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{v}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}-\frac{2}{3}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3}{4}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{1}{2}r_d(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{2}{3}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{5}{4}r_d(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{3}r_d(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{2}{3}r_d(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{1}{3}r_d(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{3}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{15}{16}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{15}{8}r_d\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+r_d\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}+\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{16}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{3}{8}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{1}{8}r_d(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{5}{4}r_d(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}}\\
&+r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{4}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{3}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{4}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{6}r_s(\boldsymbol{n}_s\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{21}{16}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+2r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{16}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}\\
&+\frac{3}{8}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{8}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{8}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{1}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{4}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3}{8} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \times \dot{\boldsymbol{\sigma}} \\
&+ \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} \\
&+ \frac{3}{4} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
&+ \frac{3}{8} r_d^{-1} r_s^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + \frac{1}{8} r_d^{-2} r_s^3 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n}_d \times \dot{\boldsymbol{\sigma}} \\
&+ \mathrm{O}(\varepsilon^2).
\end{aligned}
$$

G.6.20 Point particle self-fields

If we evaluate the fields of Section G.6.19 for $\boldsymbol{r}' = \mathbf{0}$, we obtain the generated fields for a point particle in arbitrary motion:

$$
\begin{aligned}
4\pi \boldsymbol{E}^q_{\text{point}} &= r^{-2} \boldsymbol{n} - \frac{1}{2} r^{-1} \dot{\boldsymbol{v}} - \frac{1}{2} r^{-1} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{n} + \frac{2}{3} \ddot{\boldsymbol{v}} - \frac{3}{8} \dot{\boldsymbol{v}}^2 \boldsymbol{n} + \frac{3}{4} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&\quad + \frac{3}{8} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n} - \frac{3}{8} r \dddot{\boldsymbol{v}} - \frac{15}{16} r \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}} - \frac{4}{3} r (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} - \frac{15}{16} r (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&\quad - \frac{5}{16} r (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{n} - \frac{2}{3} r (\boldsymbol{n} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{1}{8} r (\boldsymbol{n} \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n} + \frac{2}{3} r (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n} \\
&\quad + \frac{15}{16} r \dot{\boldsymbol{v}}^2 (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{n} + \mathrm{O}(r^2),
\end{aligned}
$$

$$
\begin{aligned}
4\pi \boldsymbol{E}^d_{\text{point}} &= -r^{-3} \boldsymbol{\sigma} + 3 r^{-3} (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \boldsymbol{n} + \frac{1}{2} r^{-2} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{1}{2} r^{-2} (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&\quad + \frac{1}{2} r^{-2} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n} - \frac{3}{2} r^{-2} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\boldsymbol{n} \cdot \boldsymbol{\sigma}) \boldsymbol{n} - \frac{1}{2} r^{-1} \ddot{\boldsymbol{\sigma}} + \frac{3}{8} r^{-1} \dot{\boldsymbol{v}}^2 \boldsymbol{\sigma} \\
&\quad - \frac{3}{8} r^{-1} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{1}{2} r^{-1} (\boldsymbol{n} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n} - \frac{1}{4} r^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&\quad - \frac{3}{8} r^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \boldsymbol{n} + \frac{3}{4} r^{-1} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\boldsymbol{n} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{1}{4} r^{-1} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n} \\
&\quad + \frac{9}{8} r^{-1} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \boldsymbol{n} + \frac{2}{3} \dddot{\boldsymbol{\sigma}} + \frac{3}{4} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} + \frac{5}{16} (\boldsymbol{n} \cdot \dot{\boldsymbol{v}})^3 \boldsymbol{\sigma} \\
&\quad - \frac{1}{8} (\boldsymbol{n} \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{3}{8} (\boldsymbol{n} \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{v}} + (\boldsymbol{n} \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} + \frac{3}{4} (\boldsymbol{n} \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&\quad + \frac{1}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{3}{4} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n} - \frac{1}{3} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{1}{2} (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n} - \frac{1}{8} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{15}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{15}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{16}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n} \\
&+ \frac{3}{4}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{3}{8}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{15}{16}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{3}{16}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \frac{15}{16}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} + \frac{1}{2}(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n} \\
&+ \frac{1}{8}(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} + \frac{15}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \frac{3}{8}r\ddddot{\boldsymbol{\sigma}} - \frac{15}{16}r\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{\sigma}} \\
&+ \frac{105}{128}r\dot{\boldsymbol{v}}^4\boldsymbol{\sigma} + \frac{5}{24}r\ddot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{4}{3}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{\sigma}} - \frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{\sigma}} \\
&- \frac{35}{128}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^4\boldsymbol{\sigma} - \frac{2}{3}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{5}{24}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})^2\boldsymbol{\sigma} + \frac{1}{15}r(\boldsymbol{n}\cdot\ddddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&- \frac{4}{15}r(\boldsymbol{n}\cdot\boldsymbol{\sigma})\ddddot{\boldsymbol{v}} - r(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\dddot{\boldsymbol{v}} - \frac{4}{3}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} - \frac{2}{3}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{1}{8}r(\boldsymbol{n}\cdot\ddddot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{5}{16}r(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{3}{16}r(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} - \frac{1}{4}r(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} \\
&+ \frac{3}{8}r(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{2}{3}r(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{7}{24}r(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + r(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{2}{3}r(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{7}{16}r(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{3}r(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{1}{15}r(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n} \\
&+ \frac{105}{64}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma} + r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - 2r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 3r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ \frac{15}{16}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{21}{32}r\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + r\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n} \\
&+ \frac{105}{128}r\dot{\boldsymbol{v}}^4(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} + \frac{5}{24}r\ddot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} + \frac{5}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&- \frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} - \frac{5}{2}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} - \frac{15}{8}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ 2r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{2}{3}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{9}{8}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n} \\
&+ r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n} + \frac{1}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n} \\
&- \frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n} - \frac{15}{32}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{35}{32}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{5}{32}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^3(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n} + \frac{105}{128}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^4(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n} - \frac{5}{6}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&- \frac{5}{4}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{3}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{1}{2}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{24}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\frac{5}{24}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\frac{5}{16}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{1}{16}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}-2r(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{5}{16}r(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}\\
&+\frac{5}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}+\frac{1}{3}r(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\frac{105}{32}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{3}{32}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}-\frac{105}{64}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\\
&-\frac{5}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}-\frac{5}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}+\mathrm{O}(r^2),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\boldsymbol{B}^q_{\text{point}} \;=\;& \frac{1}{2}\boldsymbol{n}\times\ddot{\boldsymbol{v}}-\frac{1}{3}r\boldsymbol{n}\times\dddot{\boldsymbol{v}}-\frac{1}{4}r\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-r\dot{\boldsymbol{v}}^2\boldsymbol{n}\times\dot{\boldsymbol{v}}-\frac{3}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\ddot{\boldsymbol{v}}\\
&-\frac{1}{2}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\mathrm{O}(r^2),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\boldsymbol{B}^d_{\text{point}} \;=\;& -r^{-2}\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{2}r^{-1}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}r^{-1}(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddot{\boldsymbol{v}}+r^{-1}(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\frac{1}{2}\boldsymbol{n}\times\ddot{\boldsymbol{\sigma}}-\frac{2}{3}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{3}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{4}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&-\frac{3}{4}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{8}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{4}(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-\frac{1}{4}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}-\frac{3}{4}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddot{\boldsymbol{v}}\\
&-\frac{3}{2}(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}-\frac{1}{2}(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}-\frac{1}{3}r\boldsymbol{n}\times\dddot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{\sigma}}+\frac{1}{4}r\ddot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}+\frac{3}{8}r\dddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{8}r\ddddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-r\dot{\boldsymbol{v}}^2\boldsymbol{n}\times\ddot{\boldsymbol{\sigma}}\\
&+\frac{15}{16}r\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{15}{16}r\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\dddot{\boldsymbol{\sigma}}\\
&+\frac{4}{3}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{2}{3}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{5}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}\times\ddot{\boldsymbol{\sigma}}\\
&+\frac{2}{3}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{2}{3}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{8}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{8}r(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddddot{\boldsymbol{v}}-\frac{1}{3}r(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\dddot{\boldsymbol{v}}\\
&-\frac{1}{2}r(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dddot{\boldsymbol{v}}-\frac{2}{3}r(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-\frac{3}{4}r(\boldsymbol{n}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}\times\ddot{\boldsymbol{v}}\\
&-\frac{1}{2}r(\boldsymbol{n}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}-\frac{2}{3}r(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}+\frac{5}{4}r(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{3}r(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddot{\boldsymbol{v}}+\frac{2}{3}r(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\frac{1}{3}r(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}\\
&-\frac{15}{16}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{\sigma}}+3r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{15}{16}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddot{\boldsymbol{v}}\\
&-\frac{15}{8}r\dot{\boldsymbol{v}}^2(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}+r\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\frac{5}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{5}{8}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}+\frac{5}{8}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}\\
&+\frac{15}{16}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\ddot{\boldsymbol{v}}+\frac{15}{8}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}\times\dot{\boldsymbol{v}}\\
&-\frac{5}{4}r(\boldsymbol{n}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\frac{5}{4}r(\boldsymbol{n}\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}\cdot\boldsymbol{\sigma})\boldsymbol{n}\times\dot{\boldsymbol{v}}+\mathrm{O}(r^2).
\end{aligned}
$$

Computing the three-divergences of these expressions, we find

$$
\begin{aligned}
\nabla\cdot\boldsymbol{E}^q_{\text{point}} &= \delta(\boldsymbol{r}),\\
\nabla\cdot\boldsymbol{E}^d_{\text{point}} &= -(\boldsymbol{\sigma}\cdot\nabla)\delta(\boldsymbol{r}),\\
\nabla\cdot\boldsymbol{B}^q_{\text{point}} &= 0,\\
\nabla\cdot\boldsymbol{B}^d_{\text{point}} &= 0.
\end{aligned}
$$

G.6.21 Redshift-weighted self-fields

Multiplying the field expressions of Section G.6.19 by the redshift factor λ, we find

$$
\begin{aligned}
\lambda\boldsymbol{E}^q_1 &= -r_d^{-1}\dot{\boldsymbol{v}}+r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\ddot{\boldsymbol{v}}-\frac{1}{2}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d+(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\
&\quad-(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{E}_2^q &= r_d^{-2}\boldsymbol{n}_d + \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{3}\ddot{\boldsymbol{v}} \\
&\quad + \frac{1}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d - \frac{3}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{3}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{E}_1^d &= -r_d^{-1}\ddot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} - r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&\quad + r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&\quad + 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&\quad + 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \dddot{\boldsymbol{\sigma}} + \frac{1}{2}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} \\
&\quad + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} \\
&\quad + \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - (\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - 2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&\quad - \frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{1}{2}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&\quad + \frac{7}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{11}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&\quad - 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&\quad + \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&\quad - 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{7}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&\quad + (\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - (\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&\quad - 3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&\quad + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&\quad + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&\quad + r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&\quad - \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{E}_2^d \;=\; &-r_d^{-2}\dot{\boldsymbol{\sigma}}-r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-2r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+6r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+r_d^{-1}\ddot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\\
&+2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}+r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}+2r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&+\frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-3r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{5}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+7r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-9r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-9r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-6r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}\dddot{\boldsymbol{\sigma}}-\frac{3}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}\\
&-\frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}-\frac{1}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{\sigma}-\frac{5}{3}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{1}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}}-3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}-\frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{2}{3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{25}{12}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+\frac{13}{4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{23}{12}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{1}{2}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{13}{8}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{3}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{13}{8}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-\frac{37}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d+4(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{13}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{37}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{21}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{25}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{27}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{11}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{17}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{11}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+3(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{33}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+16(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\\
&-r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{7}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{5}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{7}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{9}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{7}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{9}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{E}_3^d\;=\;&-r_d^{-3}\boldsymbol{\sigma}+3r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+r_d^{-2}\dot{\boldsymbol{\sigma}}+r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-6r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-3}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{2}r_d^{-3}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{2}r_d^{-1}\ddot{\boldsymbol{\sigma}}\\
&-\frac{1}{8}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}-r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}-\frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}-r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&- r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{5}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{3}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{15}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{9}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{45}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 5r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{6}\dddot{\boldsymbol{\sigma}} + \frac{1}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}} + \frac{1}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{\sigma} + \frac{11}{12}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} + \frac{3}{8}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ \frac{3}{8}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{v}} + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} + \frac{3}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{1}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{3}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&- \frac{5}{4}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{3}{4}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{3}{4}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + (\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{3}{8}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{5}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{16}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{15}{16}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{5}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{9}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{7}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{75}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{27}{16}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- \frac{19}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{7}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{15}{8}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{39}{8}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- 10(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} + \frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{16}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{16}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{5}{8}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{3}{16}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{15}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{5}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{9}{8}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{45}{16}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{5}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\lambda\boldsymbol{B}_1^q = -r_d^{-1}\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon),
$$

$$
\lambda\boldsymbol{B}_2^q = r_d^{-1}\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{2}\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon),
$$

$$
\begin{aligned}
\lambda\boldsymbol{B}_1^d = &-r_d^{-1}\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}-r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}}+\frac{1}{2}\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}+\frac{1}{2}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}+\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}+\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}+3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{3}{2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+ 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{7}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{B}_2^d \;=\; & -r_d^{-2}\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - r_d^{-2}\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + r_d^{-1}\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} \\
&+ \frac{3}{2}r_d^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} + r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&+ 3r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{5}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ 9r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{2}\boldsymbol{n}_d\times\dddot{\boldsymbol{\sigma}} - \dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} - \frac{7}{6}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
&- \frac{1}{2}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - \frac{5}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}} \\
&- \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} \\
&- \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}} - \frac{7}{6}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&- \frac{3}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}} - \frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} - \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
&- \frac{9}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{7}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + \frac{7}{2}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&+ \frac{9}{4}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{3}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} \\
&- 12(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{17}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
&- \frac{21}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{17}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}-\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-3r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{5}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{9}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda\boldsymbol{B}_3^d=\;&r_d^{-2}\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-r_d^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}-\frac{1}{2}r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&-r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{2}\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{6}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{11}{12}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dddot{\boldsymbol{v}}\\
&+\frac{1}{4}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}+\frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+\frac{3}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{3}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}-\frac{3}{2}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{9}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\frac{19}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&+\frac{3}{4}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{3}{2}r_d^{-1}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}
\end{aligned}
$$

$$- \frac{3}{4} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}} + 3 r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \times \dot{\boldsymbol{v}}$$
$$+ \mathrm{O}(\varepsilon).$$

Adding these results together, we find

$$\begin{aligned} 4\pi\lambda \boldsymbol{E}^q &= r_d^{-2} \boldsymbol{n}_d - \frac{1}{2} r_d^{-1} \dot{\boldsymbol{v}} + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{2}{3} \ddot{\boldsymbol{v}} - \frac{3}{8} \dot{\boldsymbol{v}}^2 \boldsymbol{n}_d + \frac{1}{4} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\ &\quad - \frac{1}{8} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \mathrm{O}(\varepsilon), \end{aligned}$$

$$\begin{aligned} 4\pi\lambda \boldsymbol{E}^d &= -r_d^{-3} \boldsymbol{\sigma} + 3 r_d^{-3} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{1}{2} r_d^{-2} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{1}{2} r_d^{-2} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\ &\quad - \frac{1}{2} r_d^{-3} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{3}{2} r_d^{-3} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{2} r_d^{-1} \ddot{\boldsymbol{\sigma}} \\ &\quad + \frac{3}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 \boldsymbol{\sigma} + \frac{1}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} - \frac{1}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{4} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\ &\quad - \frac{3}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\ &\quad + \frac{1}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{3}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{2}{3} \dddot{\boldsymbol{\sigma}} \\ &\quad - \frac{1}{4} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - \frac{1}{8} (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{3}{8} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{v}} + (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} \\ &\quad + \frac{3}{4} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} - \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{1}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{3}{4} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{3} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\ &\quad - \frac{1}{2} (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{1}{8} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{3}{8} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} + \frac{15}{16} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\ &\quad - \frac{3}{16} \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{1}{16} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{1}{16} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\ &\quad + \frac{1}{4} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + \frac{1}{8} (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\ &\quad + \frac{3}{8} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{4} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} + \frac{1}{4} r_d^{-1} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\ &\quad - \frac{3}{16} r_d^{-1} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{1}{16} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\ &\quad + \frac{1}{4} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{4} r_d^{-1} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{8}r_d^{-1}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{3}{16}r_d^{-1}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{8}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{8}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{3}{16}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d+\mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
4\pi\lambda\boldsymbol{B}^q = \frac{1}{2}\boldsymbol{n}_d\times\ddot{\boldsymbol{v}},
$$

$$
\begin{aligned}
4\pi\lambda\boldsymbol{B}^d =\; &-r_d^{-2}\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}+r_d^{-1}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}+\frac{1}{2}\boldsymbol{n}_d\times\ddot{\boldsymbol{\sigma}}-\frac{2}{3}\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{3}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{3}{8}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{8}(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}+\frac{1}{4}(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}}-\frac{1}{4}(\ddot{\boldsymbol{v}}\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&-\frac{1}{4}(\boldsymbol{n}_d\!\cdot\!\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_s\!\cdot\!\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}\\
&+\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{v}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{\sigma}}-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}}\\
&-\frac{1}{4}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\boldsymbol{\sigma})(\boldsymbol{n}_s\!\cdot\!\ddot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}-\frac{1}{2}r_d^{-1}r_s(\boldsymbol{n}_d\!\cdot\!\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\!\cdot\!\dot{\boldsymbol{v}})\boldsymbol{n}_d\times\dot{\boldsymbol{v}}\\
&+\mathrm{O}(\varepsilon).
\end{aligned}
$$

G.6.22 Gradients of the self-fields

We now compute the spatial gradients of the self-fields of Section G.6.19 in the direction of $\boldsymbol{\sigma}$, and multiplied by λ. (Only those expressions actually required for the radiation reaction calculations are computed.) Because of the excessive length of the resulting expressions (on the order of 14 gpages *each*), we omit terms of order ε^0 that are odd in either $\boldsymbol{n}_d$ or $\boldsymbol{n}_s$, since they do not contribute to the final equations of motion. (Odd terms of *lower* order in ε may, however, contribute to the $\boldsymbol{N}^F$ calculations, through their cross-product with $\boldsymbol{r}$.) We find

$$\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_1^q = {} & r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} + r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& - 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma} \\
& - r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + 2r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& + \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& - 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& + 2r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{2}{3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{7}{3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{4}{3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& - 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}$$

$$\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_2^q = {} & r_d^{-3}\boldsymbol{\sigma} - 3r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& - \frac{3}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-3}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} \\
& - \frac{3}{2}r_d^{-3}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{8}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma} \\
& + r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{5}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& - \frac{1}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& + \frac{9}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{9}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& - 2r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + (\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{2}{3}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& - (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{5}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& + 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}$$

$$
\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_1^d \;=\; & -r_d^{-2}\ddot{\boldsymbol{v}} - r_d^{-2}\dot{\boldsymbol{\sigma}}^2\boldsymbol{n}_d - 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d \\
& + r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} + 2r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} \\
& + r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
& + 2r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + 9r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
& + 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& + 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 15r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
& + r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 4r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
& + 4r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
& - 4r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& + 3r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - 12r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
& + 9r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + r_d^{-1}\dddot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} \\
& - \frac{5}{2}r_d^{-1}\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} + \frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{9}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
& - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d + 3r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& - r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dddot{\boldsymbol{v}} - r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& + \frac{5}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - \frac{7}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
& + r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + 3r_d^{-1}(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - 3r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
& - \frac{3}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - \frac{3}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& + 3r_d^{-1}\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
& - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - 7r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{5}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&+ \frac{9}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&- \frac{27}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - \frac{9}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 15r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&- 6r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - \frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 3r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - 3r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} \\
&- \frac{5}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + 2r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + 5r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{17}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 4r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{9}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{5}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - 3r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 9r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + 3r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{7}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 24r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&- 11r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ 9r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{31}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{19}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{27}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 18r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- 18r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{21}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{19}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} \\
&+ \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{2}r_d^{-2}r_s\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}} - \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&- \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&+ \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&- r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}} \\
&- \frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{9}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{15}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+6r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{9}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{2}\dddot{\boldsymbol{v}}-\frac{11}{4}\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}}\\
&+\frac{17}{6}\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}}-\frac{17}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}+\frac{1}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{9}{4}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&-\frac{5}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}}+\frac{43}{12}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}}+\frac{1}{2}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{11}{6}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{6}(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{1}{2}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}+\frac{9}{2}(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{17}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{11}{4}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}-\frac{3}{4}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{3}{4}\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-3\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{31}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&+\frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{33}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+\frac{17}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}}\\
&-\frac{9}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}-\frac{3}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{7}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+\frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}+\frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&+3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-4(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d+2(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&+\frac{1}{2}(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}-(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&+\frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{5}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{9}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{7}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{2}{3}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+\frac{27}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{47}{6}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+\frac{15}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{17}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-\frac{17}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d+\frac{9}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+12\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{31}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}}\\
&-2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}-3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}-16(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\\
&-11(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-9(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-25(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{39}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{33}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-33(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d+13(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{59}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{87}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-15(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-6(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{55}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-4(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{11}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-15\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+33(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}\\
&-\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}\\
&+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}\\
&+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}\\
&-r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d\\
&+r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\\
&+\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&- 3r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d \\
&+ \frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&- 2r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_2^d = {}& -2r_d^{-3}\dot{\boldsymbol{v}} + 6r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + 2r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + 6r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+ 3r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - 2r_d^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + 9r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- 30r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - 12r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ 9r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 2r_d^{-2}\ddot{\boldsymbol{v}} + 3r_d^{-2}\dot{\boldsymbol{\sigma}}^2\boldsymbol{n}_d \\
&+ 7r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} - 9r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d - 6r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - 4r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} - 3r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ 3r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{9}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + \frac{3}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ 2r_d^{-2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 4r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&- 21r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - 9r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ \frac{11}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{21}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 45r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - 8r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 24r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - 7r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&+ 9r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{23}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&- 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d - 8r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- 12r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 36r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{69}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - r_d^{-1}\dddot{\boldsymbol{v}} - \frac{3}{4}r_d^{-1}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} \\
&+\frac{5}{2}r_d^{-1}\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} - 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} - \frac{37}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&+\frac{27}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d - \frac{11}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + 3r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dddot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{1}{2}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&-3r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} + \frac{31}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&-\frac{13}{6}r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{3}{2}r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - r_d^{-1}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&-\frac{9}{2}r_d^{-1}(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d + \frac{33}{4}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{3}{4}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+\frac{3}{2}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - 4r_d^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 6r_d^{-1}\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+16r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} \\
&+12r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} + 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&-\frac{11}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - 12r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&-\frac{15}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{19}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{111}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+\frac{21}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{19}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+\frac{57}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{135}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+\frac{5}{3}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + 11r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+\frac{17}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{21}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+\frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 9r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+3r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}} + \frac{5}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&- 3r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&- 9r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{13}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{3}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{7}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{29}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 10r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ \frac{21}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - 6r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- 8r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{79}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- \frac{99}{4}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - 3r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{79}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 18r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- 64r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ \frac{37}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} \\
&- 18r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- 34r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{15}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{51}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 36r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- 42r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{369}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- \frac{51}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$+\frac{61}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}$$
$$-r_d^{-2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{3}{2}r_d^{-2}r_s\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$-\frac{7}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}+3r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{9}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d+3r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}+\frac{1}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}$$
$$+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}}+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}$$
$$-\frac{3}{2}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}-\frac{9}{4}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d$$
$$-\frac{3}{4}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$
$$-r_d^{-2}r_s(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+\frac{1}{4}r_d^{-2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}$$
$$+2r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}$$
$$+4r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}$$
$$+\frac{21}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$-12r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{9}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}$$
$$-\frac{11}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$
$$+\frac{21}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}$$
$$-\frac{45}{2}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+4r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}$$

$$
\begin{aligned}
& - 12 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& + \frac{7}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
& - \frac{9}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - \frac{9}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - \frac{23}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
& + \frac{3}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
& + 4 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
& + 4 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
& + 6 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
& - 18 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& + \frac{69}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{3} \ddddot{\boldsymbol{v}} + \dot{\boldsymbol{v}}^2 \ddot{\boldsymbol{v}} \\
& - 2 \dot{\boldsymbol{\sigma}}^2 \ddot{\boldsymbol{v}} + \frac{15}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} - (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{5}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{\sigma}} \\
& - 7 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \ddot{\boldsymbol{v}} + \frac{5}{12} (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} + \frac{11}{6} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} + (\ddot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
& + \frac{11}{12} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}} + \frac{17}{24} (\dddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} + \frac{1}{3} (\ddddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} - \frac{13}{4} (\dot{\boldsymbol{\sigma}} \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
& - \frac{31}{4} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{19}{4} \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}} + \frac{23}{8} \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
& + \frac{79}{24} \dot{\boldsymbol{v}}^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{21}{4} \dot{\boldsymbol{\sigma}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{57}{4} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
& - \frac{5}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{15}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& + \frac{15}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{\sigma}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d - \frac{39}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - 15 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \ddot{\boldsymbol{v}} + \frac{19}{4} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{27}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{19}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}-\frac{33}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&-\frac{21}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{41}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&-\frac{15}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+2(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d-(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{15}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{11}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{37}{8}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}-\frac{45}{4}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&-\frac{145}{12}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{51}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&-\frac{31}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+\frac{31}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&-\frac{15}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-18\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{57}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}}+\frac{35}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+5(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{15}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+28(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}+20(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+15(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d+43(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&+15(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{87}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-21(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+78(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&-\frac{91}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}-\frac{27}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&-\frac{69}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-51(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+33(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{21}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$+\frac{101}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}+\frac{15}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$-\frac{13}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{51}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$-\frac{167}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$-\frac{3}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d-r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$+r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}$$
$$+\frac{3}{8}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$
$$-r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}$$
$$-\frac{9}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}$$
$$-3r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$-2r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}$$
$$+6r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$-\frac{7}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}$$
$$-\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$-2r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$-2r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$-3r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$+r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$
$$+9r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$-4r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}$$

$$
\begin{aligned}
&+ 12r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- 4r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_3^d = {} & 3r_d^{-4}\boldsymbol{n}_d + 6r_d^{-4}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 15r_d^{-4}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + \frac{3}{2}r_d^{-3}\dot{\boldsymbol{v}} \\
&- 6r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - 2r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{9}{2}r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&- 3r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + 3r_d^{-3}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 9r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ 30r_d^{-3}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + 12r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- 12r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{2}r_d^{-4}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ 3r_d^{-4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{15}{2}r_d^{-4}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- r_d^{-2}\ddot{\boldsymbol{v}} - \frac{3}{8}r_d^{-2}\dot{\boldsymbol{v}}^2\boldsymbol{n}_d - \frac{3}{2}r_d^{-2}\dot{\boldsymbol{\sigma}}^2\boldsymbol{n}_d - \frac{15}{4}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}} \\
&+ \frac{45}{8}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d + 5r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{1}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} \\
&+ 2r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} + \frac{3}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - 2r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&- \frac{15}{4}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d - \frac{3}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - 2r_d^{-2}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- \frac{1}{4}r_d^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{9}{8}r_d^{-2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ 2r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{45}{4}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+ 6r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{9}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ \frac{27}{4}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{225}{8}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ 7r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 20r_d^{-2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{9}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- \frac{13}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + 3r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&+ 7r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 9r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&- 24r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{51}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{3}{8}r_d^{-1}\dddot{\boldsymbol{v}} + \frac{3}{16}r_d^{-1}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} \\
&- \frac{3}{4}r_d^{-1}\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} + \frac{5}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{v}} + \frac{75}{16}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}} \\
&- \frac{15}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3\boldsymbol{n}_d + \frac{5}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{15}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- \frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dddot{\boldsymbol{v}} - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{5}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - \frac{31}{8}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} + \frac{3}{4}r_d^{-1}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ \frac{7}{6}r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + r_d^{-1}(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{3}{4}r_d^{-1}(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ \frac{3}{2}r_d^{-1}(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d - \frac{39}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{16}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&+ \frac{19}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + 3r_d^{-1}\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- 10r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} \\
&- 5r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} - 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ 3r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{15}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ \frac{9}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{13}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{225}{16}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&- 6r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{27}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- \frac{33}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{75}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&- \frac{11}{12}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - 5r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&- \frac{19}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{17}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{9}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{45}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d \\
&- r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \ddot{\boldsymbol{v}} - \frac{3}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&+ 4 r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} - \frac{3}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&+ 3 r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&- \frac{9}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - 2 r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ 6 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{11}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{21}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{9}{2} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&+ \frac{11}{2} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{49}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ \frac{117}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d - \frac{21}{4} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- 12 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ 40 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d \\
&- \frac{15}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} \\
&+ 9 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&+ \frac{71}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{9}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&- 16 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{45}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ 24 r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{57}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
& + \frac{57}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
& - 21 r_d^{-1} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \\
& + \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} + \frac{3}{16} r_d^{-2} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& + \frac{3}{4} r_d^{-2} r_s \dot{\boldsymbol{\sigma}}^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{15}{8} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
& - \frac{5}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{45}{16} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - \frac{5}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{1}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{\sigma}} \\
& - \frac{1}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} - r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \ddot{\boldsymbol{v}} \\
& - r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{3}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
& - \frac{3}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} + r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}} \\
& + \frac{15}{8} r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d + \frac{3}{4} r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
& + r_d^{-2} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + r_d^{-2} r_s (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
& + \frac{1}{8} r_d^{-2} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{9}{16} r_d^{-2} r_s \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{\sigma}} \\
& - \frac{7}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
& - \frac{45}{8} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
& + 10 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
& - 3 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
& + \frac{9}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{27}{8} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&+ \frac{225}{16} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{7}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&+ 10 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{3}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&+ \frac{9}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{9}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&+ \frac{13}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} \\
&- \frac{3}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{n}_d \\
&- \frac{7}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{7}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{9}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ 12 r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{51}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d - \frac{1}{10} \ddddot{\boldsymbol{v}} - \frac{1}{4} \dot{\boldsymbol{v}}^2 \ddot{\boldsymbol{v}} \\
&+ \frac{1}{2} \dot{\boldsymbol{\sigma}}^2 \ddot{\boldsymbol{v}} - \frac{13}{4} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \ddot{\boldsymbol{v}} + \frac{1}{2} (\boldsymbol{n}_d \cdot \ddddot{\boldsymbol{v}}) \boldsymbol{n}_d + \frac{1}{4} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} \\
&- \frac{1}{2} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dddot{\boldsymbol{\sigma}} + \frac{11}{4} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \ddot{\boldsymbol{v}} - \frac{1}{4} (\dot{\boldsymbol{v}} \cdot \dddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{2}{3} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} \\
&- \frac{1}{2} (\ddot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{5}{12} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}} - \frac{3}{8} (\dddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{1}{5} (\ddddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ \frac{3}{4} (\dot{\boldsymbol{\sigma}} \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} + \frac{7}{2} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - 2 \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{\sigma}} - \frac{9}{8} \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
&- \frac{37}{24} \dot{\boldsymbol{v}}^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} - \frac{9}{4} \dot{\boldsymbol{\sigma}}^2 (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{13}{2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}}
\end{aligned}
$$

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\begin{aligned}
&+ (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d - 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ \frac{45}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d + \frac{13}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} \\
&- \frac{5}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{15}{8}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&- 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
&+ \frac{9}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{9}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&- \frac{25}{4}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + \frac{7}{4}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
&+ \frac{1}{2}(\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - (\boldsymbol{n}_d\cdot\ddddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d \\
&+ \frac{1}{2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 3(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 2(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d - \frac{13}{8}(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&+ \frac{9}{2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{59}{12}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
&+ \frac{21}{4}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{7}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- 7\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + 3\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
&+ 6\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + 13(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} \\
&- \frac{27}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - 2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&+ 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - 12(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&- 9(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 6(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&- 18(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - 9(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d \\
&- 12(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{51}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- 45(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d + \frac{39}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}
\end{aligned}
$$

$$+\frac{15}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{79}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$+\frac{117}{4}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$-18(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^3(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d-\frac{9}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$-\frac{23}{2}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}-\frac{7}{2}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$+(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{21}{2}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$+\frac{101}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{1}{4}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$+\frac{5}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\ddot{\boldsymbol{v}}$$
$$-\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}-\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}$$
$$-\frac{3}{8}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}-\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$
$$+\frac{1}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{\sigma}}$$
$$+\frac{3}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}$$
$$+\frac{5}{2}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d$$
$$+\frac{7}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{\sigma}$$
$$-5r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d$$
$$+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}$$
$$+\frac{3}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d$$
$$+\frac{7}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$
$$+r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}}$$
$$+\frac{9}{4}r_d^{-2}r_s^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d$$

$$
\begin{aligned}
&- r_d^{-2} r_s^2 (\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
&- 6 r_d^{-2} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})^2\boldsymbol{n}_d \\
&+ \frac{7}{2} r_d^{-2} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} \\
&- 10 r_d^{-2} r_s^2 (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{n}_d \\
&+ \frac{7}{2} r_d^{-2} r_s^2 (\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B}_1^q \;=\; & r_d^{-2}\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + 2r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& - r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} + 2r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{5}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B}_2^q \;=\; & -r_d^{-2}\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - 2r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \frac{1}{2}r_d^{-1}\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\ddot{\boldsymbol{v}} - 2r_d^{-1}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& + 2r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} - \frac{1}{6}\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{4}\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& - \frac{1}{2}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + 3(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} \\
& - (\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\times\dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon).
\end{aligned}
$$

Adding these results together, we find

$$
\begin{aligned}
4\pi\lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}^q \;=\; & r_d^{-3}\boldsymbol{\sigma} - 3r_d^{-3}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{n}_d + \frac{1}{2}r_d^{-2}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{2}r_d^{-2}(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d \\
& + \frac{1}{2}r_d^{-3}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{3}{2}r_d^{-3}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d - \frac{3}{8}r_d^{-1}\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{\sigma} + \frac{3}{4} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{3}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&- \frac{1}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{1}{4} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{3}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d + \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \boldsymbol{\sigma} - \frac{4}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{v}} \\
&- \frac{2}{3} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\lambda (\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{E}^d \;=\; & 3 r_d^{-4} \boldsymbol{n}_d + 6 r_d^{-4} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} - 15 r_d^{-4} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d - \frac{1}{2} r_d^{-3} \dot{\boldsymbol{v}} \\
&+ \frac{3}{2} r_d^{-3} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \dot{\boldsymbol{v}} + r_d^{-3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} - 3 r_d^{-3} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \\
&+ \frac{3}{2} r_d^{-4} r_s (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d + 3 r_d^{-4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{\sigma} \\
&- \frac{15}{2} r_d^{-4} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 (\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{8} r_d^{-2} \dot{\boldsymbol{v}}^2 \boldsymbol{n}_d + \frac{1}{2} r_d^{-2} \dot{\boldsymbol{\sigma}}^2 \boldsymbol{n}_d \\
&+ \frac{1}{4} r_d^{-2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{3}{8} r_d^{-2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \boldsymbol{n}_d + \frac{1}{2} r_d^{-2} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} \\
&- \frac{1}{2} r_d^{-2} (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{1}{4} r_d^{-2} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d - \frac{3}{4} r_d^{-2} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ \frac{9}{8} r_d^{-2} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d - \frac{3}{4} r_d^{-2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \dot{\boldsymbol{v}} \\
&- \frac{3}{4} r_d^{-2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{15}{8} r_d^{-2} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \boldsymbol{n}_d \\
&+ \frac{3}{2} r_d^{-2} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{n}_d + r_d^{-2} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \frac{3}{8} r_d^{-1} \dddot{\boldsymbol{v}} \\
&+ \frac{15}{16} r_d^{-1} \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}} - \frac{3}{4} r_d^{-1} \dot{\boldsymbol{\sigma}}^2 \dot{\boldsymbol{v}} - \frac{1}{16} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 \dot{\boldsymbol{v}} \\
&+ \frac{1}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \dddot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{3}{8} r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \dddot{\boldsymbol{v}} + \frac{3}{4} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} \\
&+ \frac{3}{8} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \dot{\boldsymbol{v}} - \frac{3}{4} r_d^{-1} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} - \frac{1}{2} r_d^{-1} (\ddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
&- \frac{1}{4} r_d^{-1} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{3}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d - \frac{15}{16} r_d^{-1} \dot{\boldsymbol{v}}^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \dot{\boldsymbol{v}} \\
&- \frac{9}{8} r_d^{-1} \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{3}{16} r_d^{-1} (\boldsymbol{n}_d \cdot \dot{\boldsymbol{v}})^2 (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})^2 \dot{\boldsymbol{v}}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}+\frac{1}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}}\\
&+\frac{1}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}+\frac{1}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&-\frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d-r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{v}}\\
&-\frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}+\frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{n}_d\\
&+\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{1}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{1}{2}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d+\frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&+\frac{3}{4}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}-\frac{9}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&+\frac{9}{8}r_d^{-1}\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d-\frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\\
&-\frac{3}{8}r_d^{-1}(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{n}_d\\
&-\frac{3}{4}r_d^{-1}(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{n}_d+\frac{3}{16}r_d^{-2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&-\frac{1}{4}r_d^{-2}r_s\dot{\boldsymbol{\sigma}}^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{8}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{16}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d-\frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\ddot{\boldsymbol{\sigma}}\\
&-\frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{\sigma}}+\frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&+\frac{1}{4}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}+\frac{1}{8}r_d^{-2}r_s(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\boldsymbol{n}_d\\
&+\frac{3}{8}r_d^{-2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{9}{16}r_d^{-2}r_s\dot{\boldsymbol{v}}^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d\\
&+\frac{3}{8}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\dot{\boldsymbol{v}}\\
&+\frac{3}{8}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{\sigma}\\
&-\frac{15}{16}r_d^{-2}r_s(\boldsymbol{n}_d\cdot\dot{\boldsymbol{v}})^2(\boldsymbol{n}_d\cdot\boldsymbol{\sigma})^2(\boldsymbol{n}_s\cdot\dot{\boldsymbol{v}})\boldsymbol{n}_d
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{3}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \dot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \ddot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{3}{4} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_d \cdot \ddot{\boldsymbol{\sigma}})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}}) \boldsymbol{n}_d \\
&- \frac{1}{2} r_d^{-2} r_s (\boldsymbol{n}_d \cdot \boldsymbol{\sigma})(\boldsymbol{n}_s \cdot \dot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} - \frac{4}{15} \dddot{\boldsymbol{v}} - 2 \dot{\boldsymbol{v}}^2 \ddot{\boldsymbol{v}} + \frac{4}{3} \dot{\boldsymbol{\sigma}}^2 \ddot{\boldsymbol{v}} \\
&- 2 (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}}) \dot{\boldsymbol{v}} - \frac{4}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} - \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})^2 \ddot{\boldsymbol{v}} + \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} \\
&- \frac{2}{3} (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \ddot{\boldsymbol{\sigma}} + \frac{2}{3} (\ddot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} + \frac{1}{3} (\dddot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} + \frac{2}{15} (\dddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ 2 (\dot{\boldsymbol{\sigma}} \cdot \ddot{\boldsymbol{\sigma}}) \dot{\boldsymbol{v}} + \dot{\boldsymbol{v}}^2 (\dot{\boldsymbol{v}} \cdot \dot{\boldsymbol{\sigma}}) \boldsymbol{\sigma} + \dot{\boldsymbol{v}}^2 (\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} + \frac{7}{3} (\dot{\boldsymbol{v}} \cdot \ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \boldsymbol{\sigma} \\
&+ \frac{2}{3} (\dot{\boldsymbol{v}} \cdot \boldsymbol{\sigma})(\ddot{\boldsymbol{v}} \cdot \boldsymbol{\sigma}) \dot{\boldsymbol{v}} + \mathrm{O}(\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
4\pi\lambda (\boldsymbol{\sigma} \cdot \nabla) \boldsymbol{B}^q \;=\; & -\frac{1}{2} r_d^{-1} \ddot{\boldsymbol{v}} \times \boldsymbol{\sigma} - \frac{1}{2} r_d^{-1} (\boldsymbol{n}_d \cdot \boldsymbol{\sigma}) \boldsymbol{n}_d \times \ddot{\boldsymbol{v}} + \frac{1}{3} \dddot{\boldsymbol{v}} \times \boldsymbol{\sigma} + \dot{\boldsymbol{v}}^2 \dot{\boldsymbol{v}} \times \boldsymbol{\sigma} \\
& + \mathrm{O}(\varepsilon).
\end{aligned}
$$

G.6.23 Radiation reaction self-interactions

Finally, we compute the self-interaction expressions themselves. These are obtained by means of the relations

$$
\begin{aligned}
P_{ab}^{(n)} \;&=\; \int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \, P_{ab}^{(n)}(\boldsymbol{r}_d, \boldsymbol{r}_s), \\
\boldsymbol{F}_{ab}^{(n)} \;&=\; \int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \, \boldsymbol{F}_{ab}^{(n)}(\boldsymbol{r}_d, \boldsymbol{r}_s), \\
\boldsymbol{N}_{ab}^{(n)} \;&=\; \boldsymbol{N}_{ab}^{N(n)} + \boldsymbol{N}_{ab}^{F(n)}, \\
\boldsymbol{N}_{ab}^{N(n)} \;&=\; \int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \, \boldsymbol{N}_{ab}^{N(n)}(\boldsymbol{r}_d, \boldsymbol{r}_s), \\
\boldsymbol{N}_{ab}^{F(n)} \;&=\; \int_{V_d} d^3 r_d \int_{V_s} d^3 r_s \, \boldsymbol{r}(\boldsymbol{r}_d, \boldsymbol{r}_s) \times \boldsymbol{F}_{ab}^{(n)}(\boldsymbol{r}_d, \boldsymbol{r}_s),
\end{aligned}
$$

where

$$
P_{aq}^{(n)}(\boldsymbol{r}_d, \boldsymbol{r}_s) \;=\; 0,
$$

$$
\begin{aligned}
P_{ad}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
P_{a\mu}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \dot{\boldsymbol{\sigma}}\cdot\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{aq}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{E}_n^q(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{ad}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}})\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+\dot{\boldsymbol{\sigma}}\times\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{F}_{a\mu}^{(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda(\boldsymbol{\sigma}\cdot\nabla)\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)+(\boldsymbol{\sigma}\cdot\dot{\boldsymbol{v}})\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)-\dot{\boldsymbol{\sigma}}\times\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s)\\
&\quad+\lambda\boldsymbol{\sigma}\times\boldsymbol{J}_n^a,\\
\boldsymbol{N}_{aq}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \boldsymbol{0},\\
\boldsymbol{N}_{ad}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{\sigma}\times\boldsymbol{E}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),\\
\boldsymbol{N}_{a\mu}^{N(n)}(\boldsymbol{r}_d,\boldsymbol{r}_s) &= \lambda\boldsymbol{\sigma}\times\boldsymbol{B}_n^a(\boldsymbol{r}_d,\boldsymbol{r}_s),
\end{aligned}
$$

where n is the inverse power of R of the retarded fields in question (or M for the Maxwell field of the magnetic dipole), and a and $b = q$, d or μ.

Note that the cross-interaction terms between μ and d, excepting those due to the Maxwell magnetic dipole field term, cancel by duality symmetry, and are therefore not computed.

The non-trivial self-interactions are thus

$$P_{qd}^{(1)} = -\frac{2}{3}\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})+\frac{2}{3}\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}),$$

$$P_{qd}^{(2)} = 0,$$

$$P_{q\mu}^{(1)} = 0,$$

$$P_{q\mu}^{(2)} = 0,$$

$$
\begin{aligned}
P_{dd}^{(1)} = &-\frac{2}{3}\eta_1(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})+\frac{1}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})-\frac{1}{15}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}^2-\frac{2}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})^2\\
&+\frac{2}{3}\eta_0(\dot{\boldsymbol{\sigma}}\cdot\dddot{\boldsymbol{\sigma}})+\frac{7}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})-\frac{13}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}),
\end{aligned}
$$

$$
\begin{aligned}
P_{dd}^{(2)} \;&=\; -\frac{1}{10}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + \frac{1}{15}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}}^2 + \frac{2}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})^2 + \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&\quad + \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}),
\end{aligned}
$$

$$
P_{dd}^{(3)} \;=\; 0,
$$

$$
P_{\mu\mu}^{(M)} \;=\; 0,
$$

$$
\boldsymbol{F}_{qq}^{(1)} \;=\; -\frac{2}{3}\eta_1\dot{\boldsymbol{v}} + \frac{2}{3}\eta_0\ddot{\boldsymbol{v}},
$$

$$
\boldsymbol{F}_{qq}^{(2)} \;=\; \frac{1}{6}\eta_1\dot{\boldsymbol{v}},
$$

$$
\begin{aligned}
\boldsymbol{F}_{qd}^{(1)} \;&=\; -\frac{2}{5}\eta_1\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{8}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{13}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} - \frac{29}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&\quad - \frac{29}{30}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_{qd}^{(2)} \;&=\; \eta_3'\boldsymbol{\sigma} + \frac{2}{15}\eta_1\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{2}{5}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{1}{5}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{3}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} \\
&\quad + \frac{3}{10}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_{q\mu}^{(1)} \;&=\; \frac{1}{3}\eta_2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{2}{3}\eta_1\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{2}{3}\eta_1\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{2}{3}\eta_0\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{2}\eta_0\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&\quad + \frac{27}{20}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},
\end{aligned}
$$

$$
\boldsymbol{F}_{q\mu}^{(2)} \;=\; -\frac{1}{3}\eta_2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{3}\eta_1\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{6}\eta_0\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{7}{20}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},
$$

$$
\begin{aligned}
\boldsymbol{F}_{dq}^{(1)} \;&=\; -\frac{2}{3}\eta_1\ddot{\boldsymbol{\sigma}} + \frac{1}{5}\eta_1\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{1}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{2}{3}\eta_0\dddot{\boldsymbol{\sigma}} - \frac{2}{15}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} \\
&\quad - \frac{17}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - \frac{4}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{2}{5}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$\boldsymbol{F}_{dq}^{(2)} = \frac{1}{15}\eta_1\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{1}{5}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} + \frac{7}{60}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} - \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\boldsymbol{\sigma} + \frac{7}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} + \frac{19}{60}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{1}{15}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\boldsymbol{F}_{dq}^{(3)} = -\eta_3'\boldsymbol{\sigma} + \frac{1}{60}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{\sigma}} - \frac{1}{20}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}},$$

$$\boldsymbol{F}_{\mu q}^{(1)} = -\frac{1}{3}\eta_1\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{3}\eta_1\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{3}\eta_0\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} + \frac{5}{6}\eta_0\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{3}\eta_0\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{4}{5}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{F}_{\mu q}^{(2)} = \frac{1}{3}\eta_1\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{3}\eta_0\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_0\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{13}{60}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{F}_{\mu q}^{(3)} = -\frac{1}{60}\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\begin{aligned}\boldsymbol{F}_{dd}^{(1)} = &-\frac{4}{15}\eta_2\ddot{\boldsymbol{v}} - \frac{2}{15}\eta_2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{1}{5}\eta_2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{2}{15}\eta_2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{8}{15}\eta_1\dddot{\boldsymbol{v}}\\
&+ \frac{36}{35}\eta_1\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} - \frac{14}{15}\eta_1\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} + \frac{3}{5}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - \frac{23}{210}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}}\\
&- \frac{1}{3}\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} - \frac{11}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{5}\eta_1(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{4}{5}\eta_1(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&- \frac{4}{15}\eta_1(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{149}{210}\eta_1\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{2}{5}\eta_0\ddddot{\boldsymbol{v}} - \frac{13}{5}\eta_0\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} + \frac{7}{5}\eta_0\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}}\\
&- \frac{13}{5}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{13}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} + \frac{1}{6}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{\sigma}}\\
&+ \frac{4}{5}\eta_0(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{11}{15}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} + \frac{5}{6}\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} + \frac{23}{30}\eta_0(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma}\\
&+ \frac{1}{15}\eta_0(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{11}{15}\eta_0(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{5}\eta_0(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{103}{30}\eta_0(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\\
&+ \frac{18}{35}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{12}{7}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{7}{5}\eta_0\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}\\
&+ \frac{7}{3}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{3}{7}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{5}{6}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{F}_{dd}^{(2)} = & -\frac{4}{5}\eta_3'\dot{\boldsymbol{v}} + \frac{2}{5}\eta_3'(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{4}{15}\eta_2\ddot{\boldsymbol{v}} + \frac{1}{15}\eta_2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{1}{10}\eta_2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
& - \frac{2}{15}\eta_2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{4}{15}\eta_1\dddot{\boldsymbol{v}} - \frac{26}{105}\eta_1\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} + \frac{7}{10}\eta_1\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} - \frac{1}{5}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} \\
& + \frac{5}{21}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} + \frac{1}{6}\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} + \frac{2}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{45}\eta_1(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
& + \frac{2}{15}\eta_1(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{2}{15}\eta_1(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{32}{105}\eta_1\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{2}{15}\eta_0\ddddot{\boldsymbol{v}} \\
& + \frac{71}{140}\eta_0\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} - \frac{47}{60}\eta_0\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}} + \frac{13}{28}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{6}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{\sigma}} \\
& - \frac{17}{84}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} - \frac{1}{6}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\ddot{\boldsymbol{\sigma}} - \frac{1}{12}\eta_0(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{30}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} \\
& - \frac{1}{6}\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} - \frac{1}{20}\eta_0(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{1}{12}\eta_0(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{1}{24}\eta_0(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
& - \frac{1}{15}\eta_0(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{19}{12}\eta_0(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{41}{70}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
& + \frac{29}{70}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{331}{840}\eta_0\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{557}{840}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& - \frac{73}{140}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} - \frac{247}{420}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_{dd}^{(3)} = & \frac{3}{10}\eta_3'\dot{\boldsymbol{v}} - \frac{2}{5}\eta_3'(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{1}{15}\eta_2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{1}{10}\eta_2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{4}{35}\eta_1\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} \\
& - \frac{1}{10}\eta_1\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} - \frac{1}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - \frac{13}{210}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} + \frac{1}{10}\eta_1(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
& - \frac{1}{45}\eta_1(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} + \frac{1}{30}\eta_1(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{210}\eta_1\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{13}{140}\eta_0\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} \\
& + \frac{1}{20}\eta_0\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}} + \frac{19}{140}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} + \frac{1}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{\sigma}} + \frac{1}{28}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} \\
& - \frac{1}{20}\eta_0(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{30}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} - \frac{1}{20}\eta_0(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{60}\eta_0(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
& - \frac{1}{40}\eta_0(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{3}{20}\eta_0(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{1}{14}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} \\
& - \frac{9}{70}\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{1}{168}\eta_0\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{1}{280}\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& + \frac{13}{140}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + \frac{37}{420}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},
\end{aligned}
$$

$$\boldsymbol{F}_{\mu d}^{(M)} = -3\eta_3'\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}},$$

$$\boldsymbol{F}_{\mu\mu}^{(M)} = -\frac{3}{2}\eta_3'\dot{\boldsymbol{v}},$$

$$\boldsymbol{N}_{qq}^{F(1)} = \mathbf{0},$$

$$\boldsymbol{N}_{qq}^{F(2)} = \mathbf{0},$$

$$\boldsymbol{N}_{qd}^{N(1)} = \frac{2}{3}\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{2}{3}\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{qd}^{N(2)} = -\frac{1}{6}\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{qd}^{F(1)} = -\frac{1}{6}\eta_0\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{qd}^{F(2)} = -\frac{1}{6}\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{6}\eta_0\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{6}\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{q\mu}^{N(1)} = \mathbf{0},$$

$$\boldsymbol{N}_{q\mu}^{N(2)} = \mathbf{0},$$

$$\boldsymbol{N}_{q\mu}^{F(1)} = \frac{1}{10}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{3}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\boldsymbol{N}_{q\mu}^{F(2)} = -\frac{1}{3}\eta_1\dot{\boldsymbol{\sigma}} - \frac{1}{10}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{3}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\boldsymbol{N}_{dq}^{F(1)} = \frac{1}{6}\eta_0\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} + \frac{1}{6}\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{dq}^{F(2)} = \frac{1}{6}\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{4}\eta_0\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{dq}^{F(3)} = \frac{1}{12}\eta_0\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}},$$

$$\boldsymbol{N}_{\mu q}^{F(1)} = -\frac{1}{3}\eta_0\ddot{\boldsymbol{\sigma}} + \frac{1}{10}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{1}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\boldsymbol{N}_{\mu q}^{F(2)} = -\frac{1}{3}\eta_1\dot{\boldsymbol{\sigma}} + \frac{1}{3}\eta_0\ddot{\boldsymbol{\sigma}} - \frac{2}{15}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} - \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\boldsymbol{N}_{\mu q}^{F(3)} = \frac{1}{30}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma} + \frac{1}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}},$$

$$\begin{aligned}\boldsymbol{N}_{dd}^{N(1)} = & -\frac{2}{3}\eta_1\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}} - \frac{1}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{2}{3}\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}} - \frac{2}{15}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} \\ & - \frac{1}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{4}{15}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{2}{5}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},\end{aligned}$$

$$\begin{aligned}\boldsymbol{N}_{dd}^{N(2)} = & \frac{1}{5}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{7}{60}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} - \frac{7}{30}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\ & - \frac{19}{60}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{15}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},\end{aligned}$$

$$\boldsymbol{N}_{dd}^{N(3)} = \frac{1}{60}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} + \frac{1}{20}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},$$

$$\boldsymbol{N}_{\mu\mu}^{N(M)} = \boldsymbol{0},$$

$$
\begin{aligned}
\boldsymbol{N}_{dd}^{F(1)} &= \frac{1}{10}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{20}\eta_0\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} + \frac{1}{6}\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}} + \frac{1}{6}\eta_0\dot{\boldsymbol{\sigma}}\times\ddot{\boldsymbol{\sigma}} \\
&\quad + \frac{43}{120}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} - \frac{11}{24}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{3}{10}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&\quad + \frac{19}{120}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{5}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{N}_{dd}^{F(2)} &= -\frac{1}{6}\eta_2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} + \frac{1}{3}\eta_1\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}} - \frac{2}{15}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{20}\eta_0\dot{\boldsymbol{v}}\times\ddot{\boldsymbol{v}} \\
&\quad - \frac{1}{4}\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_0\dot{\boldsymbol{\sigma}}\times\ddot{\boldsymbol{\sigma}} - \frac{59}{120}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} + \frac{97}{120}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
&\quad + \frac{13}{40}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{5}{12}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{9}{40}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma},
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{N}_{dd}^{F(3)} &= \frac{1}{6}\eta_2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} - \frac{1}{6}\eta_1\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}} + \frac{1}{30}\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{12}\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}} \\
&\quad + \frac{2}{15}\eta_0\dot{\boldsymbol{v}}^2\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} - \frac{7}{20}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{1}{40}\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
&\quad + \frac{31}{120}\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{40}\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}.
\end{aligned}
$$

$$
\boldsymbol{N}_{\mu\mu}^{F(M)} = \boldsymbol{0}.
$$

G.6.24 Final results

Adding the results of Section G.6.23 together, and defining

$$
\tilde{\mu}^2 \equiv d^2 + \mu^2,
$$

we finally find

$$
\begin{aligned}
P_{\text{self}} &= -\frac{2}{3}qd\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) - \frac{2}{3}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}}) - \frac{1}{30}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&\quad + \frac{2}{3}qd\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) + \frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{\sigma}}\cdot\dddot{\boldsymbol{\sigma}}) + \frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}}) \\
&\quad - \frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma}), \qquad \text{(G.64)}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{F}_{\text{self}} \;=\; & -\frac{3}{2}\mu^2\eta_3'\dot{\boldsymbol{v}} - \frac{1}{2}\tilde{\mu}^2\eta_3'\dot{\boldsymbol{v}} - \frac{1}{2}q^2\eta_1\dot{\boldsymbol{v}} - \frac{2}{3}qd\eta_1\ddot{\boldsymbol{\sigma}} + \frac{4}{15}\tilde{\mu}^2\eta_1\dddot{\boldsymbol{v}} + \frac{2}{3}\tilde{\mu}^2\eta_1\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}} \\
& - \frac{1}{3}\tilde{\mu}^2\eta_1\dot{\boldsymbol{\sigma}}^2\dot{\boldsymbol{v}} + \frac{1}{3}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} + \frac{1}{15}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\dot{\boldsymbol{v}} - \frac{1}{6}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} \\
& - \frac{1}{2}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{1}{5}\tilde{\mu}^2\eta_1(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{\sigma}} - \frac{19}{30}\tilde{\mu}^2\eta_1(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} \\
& - \frac{2}{15}\tilde{\mu}^2\eta_1(\dddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} - \frac{2}{5}\tilde{\mu}^2\eta_1\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{2}{3}q^2\eta_0\ddot{\boldsymbol{v}} + \frac{2}{3}qd\eta_0\dddot{\boldsymbol{\sigma}} \\
& - \frac{1}{3}qd\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}} - qd\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} - \frac{4}{15}\tilde{\mu}^2\eta_0\ddddot{\boldsymbol{v}} - 2\tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2\ddot{\boldsymbol{v}} \\
& + \frac{2}{3}\tilde{\mu}^2\eta_0\dot{\boldsymbol{\sigma}}^2\ddot{\boldsymbol{v}} - 2\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})\dot{\boldsymbol{v}} - \frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dddot{\boldsymbol{\sigma}} - \frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})^2\ddot{\boldsymbol{v}} \\
& + \frac{2}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\dddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} - \frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{\sigma}} + \frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\dot{\boldsymbol{\sigma}} \\
& + \frac{2}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{2}{3}\tilde{\mu}^2\eta_0(\dddot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \frac{2}{15}\tilde{\mu}^2\eta_0(\ddddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& + 2\tilde{\mu}^2\eta_0(\dot{\boldsymbol{\sigma}}\cdot\ddot{\boldsymbol{\sigma}})\dot{\boldsymbol{v}} + 2\tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2(\dot{\boldsymbol{v}}\cdot\dot{\boldsymbol{\sigma}})\boldsymbol{\sigma} + \tilde{\mu}^2\eta_0\dot{\boldsymbol{v}}^2(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} \\
& + \frac{5}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\ddot{\boldsymbol{v}})(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} + \frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}} \\
& - 3d\mu\eta_3'\boldsymbol{\sigma}\times\dot{\boldsymbol{\sigma}} - \frac{2}{3}q\mu\eta_1\dot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} - \frac{2}{3}q\mu\eta_1\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{4}{3}q\mu\eta_0\ddot{\boldsymbol{v}}\times\dot{\boldsymbol{\sigma}} \\
& + \frac{2}{3}q\mu\eta_0\dddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + 2q\mu\eta_0\dot{\boldsymbol{v}}^2\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}, \qquad \text{(G.65)}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{N}_{\text{self}} \;=\; & -\frac{2}{3}q\mu\eta_1\dot{\boldsymbol{\sigma}} \\
& + \frac{1}{2}qd\eta_1\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{1}{2}\tilde{\mu}^2\eta_1\boldsymbol{\sigma}\times\ddot{\boldsymbol{\sigma}} + \frac{2}{15}\tilde{\mu}^2\eta_1(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma} - \frac{2}{3}qd\eta_0\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} \\
& + \frac{2}{3}\tilde{\mu}^2\eta_0\boldsymbol{\sigma}\times\dddot{\boldsymbol{\sigma}} - \frac{1}{3}\tilde{\mu}^2\eta_0(\dot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\ddot{\boldsymbol{v}}\times\boldsymbol{\sigma} + \frac{1}{3}\tilde{\mu}^2\eta_0(\ddot{\boldsymbol{v}}\cdot\boldsymbol{\sigma})\dot{\boldsymbol{v}}\times\boldsymbol{\sigma}. \qquad \text{(G.66)}
\end{aligned}
$$

G.7 TEST3INT: Testing of 3-d integrations

```
% t3outth.txt
%
% (C) Copyright 1992, 1993, 1994 John P. Costella.
%
% LaTeX output from C program.
```

```
%  ID string: Test 3-d integrations.


This is a vector expression:
  rd n
  +rd^{-2}n
  +\half rd^{-1}a
  -\threehalves rd^{-1}(n.a)n.

This is (dummy.del) of the expression:
  dummy
  +rd^{-3}dummy
  -3rd^{-3}(n.dummy)n
  -\half rd^{-2}(n.dummy)a
  -\threehalves rd^{-2}(n.a)dummy
  -\threehalves rd^{-2}(a.dummy)n
  +\ninehalves rd^{-2}(n.a)(n.dummy)n.

This is the divergence of the vector expression:
  3
  +3rd^{-3}
  -3rd^{-3}n^2
  -\half rd^{-2}(n.a)
  -6rd^{-2}(n.a)
  +\ninehalves rd^{-2}n^2(n.a).


This is an axial expression:
  -rd^{-2}n*a
  -\half rd^{-1}a*b
  +\half rd^{-1}c*d
  +\half rd^{-1}(n.b)n*a
  +\half rd^{-1}(n.d)n*c
  +rd^{-1}(n.a)n*b.

This is (dummy.del) of the expression:
  rd^{-3}a*dummy
  +3rd^{-3}(n.dummy)n*a
  +\half rd^{-2}(n.dummy)a*b
  -\half rd^{-2}(n.dummy)c*d
  -\half rd^{-2}(n.b)a*dummy
  +\half rd^{-2}(b.dummy)n*a
  -\threehalves rd^{-2}(n.b)(n.dummy)n*a
  -\half rd^{-2}(n.d)c*dummy
  +\half rd^{-2}(d.dummy)n*c
  -\threehalves rd^{-2}(n.d)(n.dummy)n*c
  -rd^{-2}(n.a)b*dummy
  +rd^{-2}(a.dummy)n*b
  -3rd^{-2}(n.a)(n.dummy)n*b.

This is the divergence of the axial expression:
  \half rd^{-2}n.a*b
  -\half rd^{-2}n.c*d
  +\half rd^{-2}n.a*b
  +\half rd^{-2}n.c*d
```

```
  -rd^{-2}n.a*b.


This is the initial scalar expression:
  1
  +(n.a)
  +(n.b)^2
  +(n.c)^3
  +(n.d)^{-1}
  +(n.e)(n.f)
  +(n.g)(n.h)(n.i)
  +(n.j)(n.k)(n.l)(n.m)
  +(n.o)^2(n.p)(n.q)
  +(n.r)(n.s)^2(n.t)
  +(n.u)(n.v)(n.w)^2
  +(n.x)^2(n.y)^2
  +(n.a)^2(n.b)(n.z)^2
  +(n.c)^4
  +(ns.a)
  +rd^{-2}rs^2(ns.b)(ns.c)
  +(ns.d)(ns.e)(ns.f)
  +rd^{-2}rs^2(ns.g)^2
  +rd^{-2}rs^2(n.j)(n.k)(ns.h)(ns.i)
  +(n.a)^6
  +(n.c)^3(n.d)^3
  +(n.e)^2(n.f)(n.g)^2(n.h)
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)
  +rd^{-3}rs^2(ns.v)(ns.w)
  +rd^{-3}rs^2(n.a)(n.z)(ns.x)(ns.y)
  +rd^{-3}
  +rd^{-3}(n.a)(n.b)
  +rd^{-3}(n.c)(n.d)(n.e)(n.f).

This is the scalar expression with odd terms deleted:
  1
  +(n.b)^2
  +(n.e)(n.f)
  +(n.j)(n.k)(n.l)(n.m)
  +(n.o)^2(n.p)(n.q)
  +(n.r)(n.s)^2(n.t)
  +(n.u)(n.v)(n.w)^2
  +(n.x)^2(n.y)^2
  +(n.c)^4
  +rd^{-2}rs^2(ns.b)(ns.c)
  +rd^{-2}rs^2(ns.g)^2
  +rd^{-2}rs^2(n.j)(n.k)(ns.h)(ns.i)
  +(n.a)^6
  +(n.c)^3(n.d)^3
  +(n.e)^2(n.f)(n.g)^2(n.h)
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)
  +rd^{-3}rs^2(ns.v)(ns.w)
  +rd^{-3}rs^2(n.a)(n.z)(ns.x)(ns.y)
  +rd^{-3}
```

```
  +rd^{-3}(n.a)(n.b)
  +rd^{-3}(n.c)(n.d)(n.e)(n.f).

This is the integrated scalar expression:
  1
  +\third b^2
  +\third(e.f)
  +\fifteenth(j.k)(l.m)
  +\fifteenth(j.l)(k.m)
  +\fifteenth(j.m)(k.l)
  +\fifteenth o^2(p.q)
  +\twofifteenths(o.p)(o.q)
  +\fifteenth s^2(r.t)
  +\twofifteenths(r.s)(s.t)
  +\fifteenth w^2(u.v)
  +\twofifteenths(u.w)(v.w)
  +\fifteenth x^2y^2
  +\twofifteenths(x.y)^2
  +\fifth c^4
  +\fourthirds(b.c)
  +\fourthirds g^2
  +\sevenfifteenths(h.i)(j.k)
  -\f{1}{30}(h.j)(i.k)
  -\f{1}{30}(h.k)(i.j)
  +\f{1}{7}a^6
  +\f{2}{35}(c.d)^3
  +\f{3}{35}c^2d^2(c.d)
  +\f{1}{105}e^2g^2(f.h)
  +\f{2}{105}(e.g)^2(f.h)
  +\f{2}{105}e^2(f.g)(g.h)
  +\f{2}{105}g^2(e.f)(e.h)
  +\f{4}{105}(e.f)(e.g)(g.h)
  +\f{4}{105}(e.g)(e.h)(f.g)
  +\f{1}{105}(i.j)(k.l)(m.o)
  +\f{1}{105}(i.j)(k.m)(l.o)
  +\f{1}{105}(i.j)(k.o)(l.m)
  +\f{1}{105}(i.k)(j.l)(m.o)
  +\f{1}{105}(i.k)(j.m)(l.o)
  +\f{1}{105}(i.k)(j.o)(l.m)
  +\f{1}{105}(i.l)(j.k)(m.o)
  +\f{1}{105}(i.l)(j.m)(k.o)
  +\f{1}{105}(i.l)(j.o)(k.m)
  +\f{1}{105}(i.m)(j.k)(l.o)
  +\f{1}{105}(i.m)(j.l)(k.o)
  +\f{1}{105}(i.m)(j.o)(k.l)
  +\f{1}{105}(i.o)(j.k)(l.m)
  +\f{1}{105}(i.o)(j.l)(k.m)
  +\f{1}{105}(i.o)(j.m)(k.l)
  +\f{2}{21}(p.q)(r.s)(t.u)
  -\f{1}{210}(p.q)(r.t)(s.u)
  -\f{1}{210}(p.q)(r.u)(s.t)
  +\f{2}{21}(p.r)(q.s)(t.u)
  -\f{1}{210}(p.r)(q.t)(s.u)
  -\f{1}{210}(p.r)(q.u)(s.t)
  +\f{2}{21}(p.s)(q.r)(t.u)
```

```
  -\f{1}{210}(p.s)(q.t)(r.u)
  -\f{1}{210}(p.s)(q.u)(r.t)
  -\f{1}{210}(p.t)(q.r)(s.u)
  -\f{1}{210}(p.t)(q.s)(r.u)
  -\f{1}{210}(p.t)(q.u)(r.s)
  -\f{1}{210}(p.u)(q.r)(s.t)
  -\f{1}{210}(p.u)(q.s)(r.t)
  -\f{1}{210}(p.u)(q.t)(r.s)
  +rd^{-11}(v.w)
  -\third rd^{-1}(v.w)
  +\third rd^{-11}(a.z)(x.y)
  -\fifteenth rd^{-1}(a.x)(y.z)
  -\fifteenth rd^{-1}(a.y)(x.z)
  -\fifteenth rd^{-1}(a.z)(x.y)
  -7eta'
  +3eta''
  -\f{8}{3}eta'(a.b)
  +eta''(a.b)
  -\f{43}{75}eta'(c.d)(e.f)
  -\f{43}{75}eta'(c.e)(d.f)
  -\f{43}{75}eta'(c.f)(d.e)
  +\fifth eta''(c.d)(e.f)
  +\fifth eta''(c.e)(d.f)
  +\fifth eta''(c.f)(d.e).


This is the initial vector expression:
  a
  +n
  +(n.b)c
  +(n.d)^2e
  +(n.f)^3g
  +(n.h)^{-1}i
  +(n.j)(n.k)l
  +(n.m)(n.o)(n.p)q
  +(n.r)(n.s)(n.t)(n.u)v
  +(n.w)^2(n.x)(n.y)z
  +(n.a)(n.b)^2(n.c)d
  +(n.e)(n.f)(n.g)^2h
  +(n.i)^2(n.j)^2k
  +(n.l)^2(n.m)(n.o)^2p
  +(n.q)^4r
  +(n.s)n
  +(n.t)^2n
  +(n.u)^3n
  +(n.v)^4n
  +(n.w)(n.x)n
  +(n.a)(n.y)(n.z)n
  +(n.b)(n.c)(n.d)(n.e)n
  +(n.f)^2(n.g)n
  +(n.h)(n.i)^2n
  +ns
  +rd^{-2}rs^2(ns.j)ns
  +(ns.k)(ns.l)ns
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)ns
```

```
  +(n.q)(n.r)(ns.s)(ns.t)ns
  +(ns.u)v
  +rd^{-2}rs^2(ns.w)(ns.x)y
  +(ns.a)(ns.b)(ns.z)c
  +rd^{-2}rs^2(n.d)(n.e)(ns.f)(ns.g)h
  +(n.b)^6a
  +(n.c)^3(n.d)^3a
  +(n.e)^2(n.f)(n.g)^2(n.h)a
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)a
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)a
  +(n.p)^5n
  +(n.q)^3(n.r)^2n
  +(n.s)^2(n.t)(n.u)(n.v)n
  +(n.a)(n.w)(n.x)(n.y)(n.z)n
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)n
  +rd^{-3}rs^2(ns.v)(ns.w)b
  +rd^{-3}rs^2(n.a)(n.z)(ns.x)(ns.y)b
  +rd^{-3}rs^2(ns.b)ns
  +rd^{-3}rs^2(n.d)(n.e)(ns.c)ns.

This is the vector expression with odd terms deleted:
  a
  +(n.d)^2e
  +(n.j)(n.k)l
  +(n.r)(n.s)(n.t)(n.u)v
  +(n.w)^2(n.x)(n.y)z
  +(n.a)(n.b)^2(n.c)d
  +(n.e)(n.f)(n.g)^2h
  +(n.i)^2(n.j)^2k
  +(n.q)^4r
  +(n.s)n
  +(n.u)^3n
  +(n.a)(n.y)(n.z)n
  +(n.f)^2(n.g)n
  +(n.h)(n.i)^2n
  +rd^{-2}rs^2(ns.j)ns
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)ns
  +rd^{-2}rs^2(ns.w)(ns.x)y
  +rd^{-2}rs^2(n.d)(n.e)(ns.f)(ns.g)h
  +(n.b)^6a
  +(n.c)^3(n.d)^3a
  +(n.e)^2(n.f)(n.g)^2(n.h)a
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)a
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)a
  +(n.p)^5n
  +(n.q)^3(n.r)^2n
  +(n.s)^2(n.t)(n.u)(n.v)n
  +(n.a)(n.w)(n.x)(n.y)(n.z)n
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)n
  +rd^{-3}rs^2(ns.v)(ns.w)b
  +rd^{-3}rs^2(n.a)(n.z)(ns.x)(ns.y)b
  +rd^{-3}rs^2(ns.b)ns
  +rd^{-3}rs^2(n.d)(n.e)(ns.c)ns.

This is the integrated vector expression:
```

```
a
+\third d^2e
+\third(j.k)l
+\fifteenth(r.s)(t.u)v
+\fifteenth(r.t)(s.u)v
+\fifteenth(r.u)(s.t)v
+\fifteenth w^2(x.y)z
+\twofifteenths(w.x)(w.y)z
+\fifteenth b^2(a.c)d
+\twofifteenths(a.b)(b.c)d
+\fifteenth g^2(e.f)h
+\twofifteenths(e.g)(f.g)h
+\fifteenth i^2j^2k
+\twofifteenths(i.j)^2k
+\fifth q^4r
+\third s
+\fifth u^2u
+\fifteenth(a.y)z
+\fifteenth(a.z)y
+\fifteenth(y.z)a
+\fifteenth f^2g
+\twofifteenths(f.g)f
+\fifteenth i^2h
+\twofifteenths(h.i)i
+\fourthirds j
+\sevenfifteenths(m.o)p
-\f{1}{30}(m.p)o
-\f{1}{30}(o.p)m
+\fourthirds(w.x)y
+\sevenfifteenths(d.e)(f.g)h
-\f{1}{30}(d.f)(e.g)h
-\f{1}{30}(d.g)(e.f)h
+\f{1}{7}b^6a
+\f{2}{35}(c.d)^3a
+\f{3}{35}c^2d^2(c.d)a
+\f{1}{105}e^2g^2(f.h)a
+\f{2}{105}(e.g)^2(f.h)a
+\f{2}{105}e^2(f.g)(g.h)a
+\f{2}{105}g^2(e.f)(e.h)a
+\f{4}{105}(e.f)(e.g)(g.h)a
+\f{4}{105}(e.g)(e.h)(f.g)a
+\f{1}{105}(i.j)(k.l)(m.o)a
+\f{1}{105}(i.j)(k.m)(l.o)a
+\f{1}{105}(i.j)(k.o)(l.m)a
+\f{1}{105}(i.k)(j.l)(m.o)a
+\f{1}{105}(i.k)(j.m)(l.o)a
+\f{1}{105}(i.k)(j.o)(l.m)a
+\f{1}{105}(i.l)(j.k)(m.o)a
+\f{1}{105}(i.l)(j.m)(k.o)a
+\f{1}{105}(i.l)(j.o)(k.m)a
+\f{1}{105}(i.m)(j.k)(l.o)a
+\f{1}{105}(i.m)(j.l)(k.o)a
+\f{1}{105}(i.m)(j.o)(k.l)a
+\f{1}{105}(i.o)(j.k)(l.m)a
+\f{1}{105}(i.o)(j.l)(k.m)a
```

```
+\f{1}{105}(i.o)(j.m)(k.l)a
+\f{2}{21}(p.q)(r.s)(t.u)a
-\f{1}{210}(p.q)(r.t)(s.u)a
-\f{1}{210}(p.q)(r.u)(s.t)a
+\f{2}{21}(p.r)(q.s)(t.u)a
-\f{1}{210}(p.r)(q.t)(s.u)a
-\f{1}{210}(p.r)(q.u)(s.t)a
+\f{2}{21}(p.s)(q.r)(t.u)a
-\f{1}{210}(p.s)(q.t)(r.u)a
-\f{1}{210}(p.s)(q.u)(r.t)a
-\f{1}{210}(p.t)(q.r)(s.u)a
-\f{1}{210}(p.t)(q.s)(r.u)a
-\f{1}{210}(p.t)(q.u)(r.s)a
-\f{1}{210}(p.u)(q.r)(s.t)a
-\f{1}{210}(p.u)(q.s)(r.t)a
-\f{1}{210}(p.u)(q.t)(r.s)a
+\f{1}{7}p^4p
+\f{1}{35}q^2r^2q
+\f{2}{35}(q.r)^2q
+\f{2}{35}q^2(q.r)r
+\f{1}{105}s^2(t.u)v
+\f{1}{105}s^2(t.v)u
+\f{1}{105}s^2(u.v)t
+\f{2}{105}(s.t)(s.u)v
+\f{2}{105}(s.t)(s.v)u
+\f{2}{105}(s.t)(u.v)s
+\f{2}{105}(s.u)(s.v)t
+\f{2}{105}(s.u)(t.v)s
+\f{2}{105}(s.v)(t.u)s
+\f{1}{105}(a.w)(x.y)z
+\f{1}{105}(a.w)(x.z)y
+\f{1}{105}(a.w)(y.z)x
+\f{1}{105}(a.x)(w.y)z
+\f{1}{105}(a.x)(w.z)y
+\f{1}{105}(a.x)(y.z)w
+\f{1}{105}(a.y)(w.x)z
+\f{1}{105}(a.y)(w.z)x
+\f{1}{105}(a.y)(x.z)w
+\f{1}{105}(a.z)(w.x)y
+\f{1}{105}(a.z)(w.y)x
+\f{1}{105}(a.z)(x.y)w
+\f{1}{105}(w.x)(y.z)a
+\f{1}{105}(w.y)(x.z)a
+\f{1}{105}(w.z)(x.y)a
-\f{1}{210}(b.c)(d.e)f
-\f{1}{210}(b.c)(d.f)e
+\f{2}{21}(b.c)(e.f)d
-\f{1}{210}(b.d)(c.e)f
-\f{1}{210}(b.d)(c.f)e
+\f{2}{21}(b.d)(e.f)c
-\f{1}{210}(b.e)(c.d)f
-\f{1}{210}(b.e)(c.f)d
-\f{1}{210}(b.e)(d.f)c
-\f{1}{210}(b.f)(c.d)e
-\f{1}{210}(b.f)(c.e)d
```

```
  -\f{1}{210}(b.f)(d.e)c
  +\f{2}{21}(c.d)(e.f)b
  -\f{1}{210}(c.e)(d.f)b
  -\f{1}{210}(c.f)(d.e)b
  +rd^{-11}(v.w)b
  -\third rd^{-1}(v.w)b
  +\third rd^{-11}(a.z)(x.y)b
  -\fifteenth rd^{-1}(a.x)(y.z)b
  -\fifteenth rd^{-1}(a.y)(x.z)b
  -\fifteenth rd^{-1}(a.z)(x.y)b
  +rd^{-11}b
  -\third rd^{-1}b
  +\third rd^{-11}(d.e)c
  -\fifteenth rd^{-1}(c.d)e
  -\fifteenth rd^{-1}(c.e)d
  -\fifteenth rd^{-1}(d.e)c.


This is the initial axial expression:
  a*b
  +n*c
  +(n.d)e*f
  +(n.g)^2h*i
  +(n.j)^3k*l
  +(n.m)^{-1}o*p
  +(n.q)(n.r)s*t
  +(n.u)(n.v)(n.w)x*y
  +(n.a)(n.b)(n.c)(n.z)d*e
  +(n.f)^2(n.g)(n.h)i*j
  +(n.k)(n.l)^2(n.m)o*p
  +(n.q)(n.r)(n.s)^2t*u
  +(n.v)^2(n.w)^2x*y
  +(n.a)(n.b)^2(n.z)^2c*d
  +(n.e)^4f*g
  +(n.h)n*i
  +(n.j)^2n*k
  +(n.l)^3n*m
  +(n.o)^4n*p
  +(n.q)(n.r)n*s
  +(n.t)(n.u)(n.v)n*w
  +(n.a)(n.x)(n.y)(n.z)n*b
  +(n.c)^2(n.d)n*e
  +(n.f)(n.g)^2n*h
  +(n.h)a'*n
  +(n.j)^2a'*n
  +(n.l)^3a'*n
  +(n.o)^4a'*n
  +(n.q)(n.r)a'*n
  +(n.t)(n.u)(n.v)a'*n
  +(n.a)(n.x)(n.y)(n.z)a'*n
  +(n.c)^2(n.d)a'*n
  +(n.f)(n.g)^2a'*n
  +(ns.h)i*j
  +rd^{-2}rs^2(ns.k)(ns.l)m*o
  +(ns.p)(ns.q)(ns.r)s*t
```

```
  +rd^{-2}rs^2(n.u)(n.v)(ns.w)(ns.x)y*z
  +ns*a
  +rd^{-2}rs^2(ns.j)ns*a
  +(ns.k)(ns.l)ns*a
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)ns*a
  +(n.q)(n.r)(ns.s)(ns.t)ns*a
  +a'*ns
  +rd^{-2}rs^2(ns.j)a'*ns
  +(ns.k)(ns.l)a'*ns
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)a'*ns
  +(n.q)(n.r)(ns.s)(ns.t)a'*ns
  +(n.b)^6a*b
  +(n.c)^3(n.d)^3a*b
  +(n.e)^2(n.f)(n.g)^2(n.h)a*b
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)a*b
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)a*b
  +(n.p)^5n*b
  +(n.q)^3(n.r)^2n*b
  +(n.s)^2(n.t)(n.u)(n.v)n*b
  +(n.a)(n.w)(n.x)(n.y)(n.z)n*b
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)n*b
  +(n.p)^5a'*n
  +(n.q)^3(n.r)^2a'*n
  +(n.s)^2(n.t)(n.u)(n.v)a'*n
  +(n.a)(n.w)(n.x)(n.y)(n.z)a'*n
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)a'*n.

This is the axial expression with odd terms deleted:
  a*b
  +(n.g)^2h*i
  +(n.q)(n.r)s*t
  +(n.a)(n.b)(n.c)(n.z)d*e
  +(n.f)^2(n.g)(n.h)i*j
  +(n.k)(n.l)^2(n.m)o*p
  +(n.q)(n.r)(n.s)^2t*u
  +(n.v)^2(n.w)^2x*y
  +(n.e)^4f*g
  +(n.h)n*i
  +(n.l)^3n*m
  +(n.t)(n.u)(n.v)n*w
  +(n.c)^2(n.d)n*e
  +(n.f)(n.g)^2n*h
  +(n.h)a'*n
  +(n.l)^3a'*n
  +(n.t)(n.u)(n.v)a'*n
  +(n.c)^2(n.d)a'*n
  +(n.f)(n.g)^2a'*n
  +rd^{-2}rs^2(ns.k)(ns.l)m*o
  +rd^{-2}rs^2(n.u)(n.v)(ns.w)(ns.x)y*z
  +rd^{-2}rs^2(ns.j)ns*a
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)ns*a
  +rd^{-2}rs^2(ns.j)a'*ns
  +rd^{-2}rs^2(n.m)(n.o)(ns.p)a'*ns
  +(n.b)^6a*b
  +(n.c)^3(n.d)^3a*b
```

```
  +(n.e)^2(n.f)(n.g)^2(n.h)a*b
  +(n.i)(n.j)(n.k)(n.l)(n.m)(n.o)a*b
  +rd^{-2}rs^2(n.p)(n.q)(n.r)(n.s)(ns.t)(ns.u)a*b
  +(n.p)^5n*b
  +(n.q)^3(n.r)^2n*b
  +(n.s)^2(n.t)(n.u)(n.v)n*b
  +(n.a)(n.w)(n.x)(n.y)(n.z)n*b
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)n*b
  +(n.p)^5a'*n
  +(n.q)^3(n.r)^2a'*n
  +(n.s)^2(n.t)(n.u)(n.v)a'*n
  +(n.a)(n.w)(n.x)(n.y)(n.z)a'*n
  +rd^{-2}rs^2(n.b)(n.c)(n.d)(ns.e)(ns.f)a'*n.

This is the integrated axial expression:
  a*b
  +\third g^2h*i
  +\third(q.r)s*t
  +\fifteenth(a.b)(c.z)d*e
  +\fifteenth(a.c)(b.z)d*e
  +\fifteenth(a.z)(b.c)d*e
  +\fifteenth f^2(g.h)i*j
  +\twofifteenths(f.g)(f.h)i*j
  +\fifteenth l^2(k.m)o*p
  +\twofifteenths(k.l)(l.m)o*p
  +\fifteenth s^2(q.r)t*u
  +\twofifteenths(q.s)(r.s)t*u
  +\fifteenth v^2w^2x*y
  +\twofifteenths(v.w)^2x*y
  +\fifth e^4f*g
  +\third h*i
  +\fifth l^2l*m
  +\fifteenth(t.u)v*w
  +\fifteenth(t.v)u*w
  +\fifteenth(u.v)t*w
  +\fifteenth c^2d*e
  +\twofifteenths(c.d)c*e
  +\fifteenth g^2f*h
  +\twofifteenths(f.g)g*h
  +\third a'*h
  +\fifth l^2a'*l
  +\fifteenth(t.u)a'*v
  +\fifteenth(t.v)a'*u
  +\fifteenth(u.v)a'*t
  +\fifteenth c^2a'*d
  +\twofifteenths(c.d)a'*c
  +\fifteenth g^2a'*f
  +\twofifteenths(f.g)a'*g
  +\fourthirds(k.l)m*o
  +\sevenfifteenths(u.v)(w.x)y*z
  -\f{1}{30}(u.w)(v.x)y*z
  -\f{1}{30}(u.x)(v.w)y*z
  -\fourthirds a*j
  -\sevenfifteenths(m.o)a*p
  +\f{1}{30}(m.p)a*o
```

```
+\f{1}{30}(o.p)a*m
+\fourthirds a'*j
+\sevenfifteenths(m.o)a'*p
-\f{1}{30}(m.p)a'*o
-\f{1}{30}(o.p)a'*m
+\f{1}{7}b^6a*b
+\f{2}{35}(c.d)^3a*b
+\f{3}{35}c^2d^2(c.d)a*b
+\f{1}{105}e^2g^2(f.h)a*b
+\f{2}{105}(e.g)^2(f.h)a*b
+\f{2}{105}e^2(f.g)(g.h)a*b
+\f{2}{105}g^2(e.f)(e.h)a*b
+\f{4}{105}(e.f)(e.g)(g.h)a*b
+\f{4}{105}(e.g)(e.h)(f.g)a*b
+\f{1}{105}(i.j)(k.l)(m.o)a*b
+\f{1}{105}(i.j)(k.m)(l.o)a*b
+\f{1}{105}(i.j)(k.o)(l.m)a*b
+\f{1}{105}(i.k)(j.l)(m.o)a*b
+\f{1}{105}(i.k)(j.m)(l.o)a*b
+\f{1}{105}(i.k)(j.o)(l.m)a*b
+\f{1}{105}(i.l)(j.k)(m.o)a*b
+\f{1}{105}(i.l)(j.m)(k.o)a*b
+\f{1}{105}(i.l)(j.o)(k.m)a*b
+\f{1}{105}(i.m)(j.k)(l.o)a*b
+\f{1}{105}(i.m)(j.l)(k.o)a*b
+\f{1}{105}(i.m)(j.o)(k.l)a*b
+\f{1}{105}(i.o)(j.k)(l.m)a*b
+\f{1}{105}(i.o)(j.l)(k.m)a*b
+\f{1}{105}(i.o)(j.m)(k.l)a*b
+\f{2}{21}(p.q)(r.s)(t.u)a*b
-\f{1}{210}(p.q)(r.t)(s.u)a*b
-\f{1}{210}(p.q)(r.u)(s.t)a*b
+\f{2}{21}(p.r)(q.s)(t.u)a*b
-\f{1}{210}(p.r)(q.t)(s.u)a*b
-\f{1}{210}(p.r)(q.u)(s.t)a*b
+\f{2}{21}(p.s)(q.r)(t.u)a*b
-\f{1}{210}(p.s)(q.t)(r.u)a*b
-\f{1}{210}(p.s)(q.u)(r.t)a*b
-\f{1}{210}(p.t)(q.r)(s.u)a*b
-\f{1}{210}(p.t)(q.s)(r.u)a*b
-\f{1}{210}(p.t)(q.u)(r.s)a*b
-\f{1}{210}(p.u)(q.r)(s.t)a*b
-\f{1}{210}(p.u)(q.s)(r.t)a*b
-\f{1}{210}(p.u)(q.t)(r.s)a*b
-\f{1}{7}p^4b*p
-\f{1}{35}q^2r^2b*q
-\f{2}{35}(q.r)^2b*q
-\f{2}{35}q^2(q.r)b*r
-\f{1}{105}s^2(t.u)b*v
-\f{1}{105}s^2(t.v)b*u
-\f{1}{105}s^2(u.v)b*t
-\f{2}{105}(s.t)(s.u)b*v
-\f{2}{105}(s.t)(s.v)b*u
-\f{2}{105}(s.t)(u.v)b*s
-\f{2}{105}(s.u)(s.v)b*t
```

```
-\f{2}{105}(s.u)(t.v)b*s
-\f{2}{105}(s.v)(t.u)b*s
-\f{1}{105}(a.w)(x.y)b*z
-\f{1}{105}(a.w)(x.z)b*y
-\f{1}{105}(a.w)(y.z)b*x
-\f{1}{105}(a.x)(w.y)b*z
-\f{1}{105}(a.x)(w.z)b*y
-\f{1}{105}(a.x)(y.z)b*w
-\f{1}{105}(a.y)(w.x)b*z
-\f{1}{105}(a.y)(w.z)b*x
-\f{1}{105}(a.y)(x.z)b*w
-\f{1}{105}(a.z)(w.x)b*y
-\f{1}{105}(a.z)(w.y)b*x
-\f{1}{105}(a.z)(x.y)b*w
+\f{1}{105}(w.x)(y.z)a*b
+\f{1}{105}(w.y)(x.z)a*b
+\f{1}{105}(w.z)(x.y)a*b
+\f{1}{210}(b.c)(d.e)b*f
+\f{1}{210}(b.c)(d.f)b*e
-\f{2}{21}(b.c)(e.f)b*d
+\f{1}{210}(b.d)(c.e)b*f
+\f{1}{210}(b.d)(c.f)b*e
-\f{2}{21}(b.d)(e.f)b*c
+\f{1}{210}(b.e)(c.d)b*f
+\f{1}{210}(b.e)(c.f)b*d
+\f{1}{210}(b.e)(d.f)b*c
+\f{1}{210}(b.f)(c.d)b*e
+\f{1}{210}(b.f)(c.e)b*d
+\f{1}{210}(b.f)(d.e)b*c
+\f{1}{7}p^4a'*p
+\f{1}{35}q^2r^2a'*q
+\f{2}{35}(q.r)^2a'*q
+\f{2}{35}q^2(q.r)a'*r
+\f{1}{105}s^2(t.u)a'*v
+\f{1}{105}s^2(t.v)a'*u
+\f{1}{105}s^2(u.v)a'*t
+\f{2}{105}(s.t)(s.u)a'*v
+\f{2}{105}(s.t)(s.v)a'*u
+\f{2}{105}(s.t)(u.v)a'*s
+\f{2}{105}(s.u)(s.v)a'*t
+\f{2}{105}(s.u)(t.v)a'*s
+\f{2}{105}(s.v)(t.u)a'*s
+\f{1}{105}(a.w)(x.y)a'*z
+\f{1}{105}(a.w)(x.z)a'*y
+\f{1}{105}(a.w)(y.z)a'*x
+\f{1}{105}(a.x)(w.y)a'*z
+\f{1}{105}(a.x)(w.z)a'*y
+\f{1}{105}(a.x)(y.z)a'*w
+\f{1}{105}(a.y)(w.x)a'*z
+\f{1}{105}(a.y)(w.z)a'*x
+\f{1}{105}(a.y)(x.z)a'*w
+\f{1}{105}(a.z)(w.x)a'*y
+\f{1}{105}(a.z)(w.y)a'*x
+\f{1}{105}(a.z)(x.y)a'*w
+\f{1}{105}(w.x)(y.z)a'*a
```

```
  +\f{1}{105}(w.y)(x.z)a'*a
  +\f{1}{105}(w.z)(x.y)a'*a
  -\f{1}{210}(b.c)(d.e)a'*f
  -\f{1}{210}(b.c)(d.f)a'*e
  +\f{2}{21}(b.c)(e.f)a'*d
  -\f{1}{210}(b.d)(c.e)a'*f
  -\f{1}{210}(b.d)(c.f)a'*e
  +\f{2}{21}(b.d)(e.f)a'*c
  -\f{1}{210}(b.e)(c.d)a'*f
  -\f{1}{210}(b.e)(c.f)a'*d
  -\f{1}{210}(b.e)(d.f)a'*c
  -\f{1}{210}(b.f)(c.d)a'*e
  -\f{1}{210}(b.f)(c.e)a'*d
  -\f{1}{210}(b.f)(d.e)a'*c
  +\f{2}{21}(c.d)(e.f)a'*b
  -\f{1}{210}(c.e)(d.f)a'*b
  -\f{1}{210}(c.f)(d.e)a'*b.


This is the initial triple expression:
  a.b*c
  +n.d*e
  +(n.i)n.g*h
  +(n.j)n.j*k
  +(n.o)n.o*p.

This is the triple expression with odd terms deleted:
  a.b*c
  +(n.i)n.g*h
  +(n.j)n.j*k
  +(n.o)n.o*p.

This is the integrated triple expression:
  a.b*c
  +\third g.h*i.

%
%  (C) Copyright 1992, 1993, 1994 John P. Costella.
%
%  End of file.
```

G.8 CHECKRS: Checking of inner integrals

```
%  crsoutth.txt
%
%  (C) Copyright 1992, 1993, 1994 John P. Costella.
%
%  Output from C program.
%  ID string: Checking inner integrals.
%

Steps = 10:
  Computations  = 10000
```

```
  Integral      = 4.048742
  Normalisation = 1.011783

Steps = 20:
  Computations  = 160000
  Integral      = 4.019799
  Normalisation = 1.003951

Steps = 30:
  Computations  = 810000
  Integral      = 4.011310
  Normalisation = 1.002102

Steps = 40:
  Computations  = 2560000
  Integral      = 4.007539
  Normalisation = 1.001347

Steps = 50:
  Computations  = 6250000
  Integral      = 4.005485
  Normalisation = 1.000955

%
%  (C) Copyright 1992, 1993, 1994 John P. Costella.
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%  End of file.
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It doesn't matter whether you succeed or fail, John Paul, as long as you know you have done your best.

—— John Joseph Costella (1979)